ISOTOPIC, GENOTOPIC AND HYPERSTRUCTURAL METHODS IN THEORETICAL BIOLOGY

Ruggero Maria Santilli
Institute for Basic Research
P. O. Box 1577
Palm Harbor, FL 34682, U.S.A.

THE PRICE OF NOVELTY

..... one of the most influential symphonies even composed, Beethoven's 9-th Symphony in D-minor, was not understood by the public. During its first two performances in Wien, in May 1824, the theater was half-empty and most left before the end.....

PREFACE

Introductory comments. This monograph grew out of my conviction, originating since the time of my high school studies, that biological systems are structurally more complex than physical systems, thus requiring more general mathematical methods for their treatment in a more effective quantitative form.

Subsequently, this monograph was stimulated by my uneasiness in seeing that contemporary theoretical biology is treated via essentially the same methods used in physics, thus implying their *de facto* identity.

More recently, the writing of this monograph originated the various seminal talks I heard at the *International Workshop on New Frontiers in Theoretical Biology*, held at our Institute, Castle Prince Pignatelli, Molise, Italy on August 1995, and, in particular, by the illuminating talk on the problem of irreversibility by H. Tributsch, Director of the Hahn–Meitner Institute in Berlin, Germany (see the Proceedings [a]).

A simple comparative inspection of biological and physical systems confirms the structurally broader character of the former over the latter. A first, most evident difference is the intrinsically *nonconservative* character of biological systems (because they grow or decay in time), as compared to the notorious *conservative* character of physical systems. To comply with evidence, the former require quantitative representations of *time–rate–of–variations* of their characteristics (such as size, weight, etc.), which admit as particular case the familiar *conservation laws* at the foundation of contemporary physics. It is then evident that methods established for the latter should not expected to be equally effective for the former.

Numerous other differences also exist. For instance, biological systems are

simply cannot be assumed to be exactly valid for the description of the constituents of all irreversible macroscopic systems.

Needless to say, the *approximate* validity of quantum mechanics under the conditions considered is beyond question. The point is that such an approximate character is per se evidence of the existence of non-quantum-mechanical effects in biological structures which are precisely the origin of irreversibility.

This occurrence is deeply linked to the preceding ones. All known action-at-a-distance interactions are reversible. The assumption that biological systems are solely treatable with potential interactions is then incompatible with its irreversible behaviour.

Vice-versa, the admission of additional contact, zero-range interactions in the interior of biological systems implies their irreversibility because said interactions are structurally irreversible for various known reasons, e.g., because not representable with a Lagrangian or a Hamiltonian.

Similarly, the use of quantum mechanics in theoretical biology implies that macroscopic systems are not only reversible but also conservative, which is grossly contrary to evidence. The admission of time-rate-of-variations of total characteristics requires their representation *beginning* at the level of each biological constituents. The recovering of irreversibility is then consequential because time-rate-of-variations are known to be irreversible (technically, they violate the theorem of detailed balancing).

The above occurrences establish beyond credible doubts that biological systems require methods structurally more general than the quantum mechanical methods of contemporary physics. In particular, the generalized methods must be intrinsically nonconservative, irreversible and nonlocal-integral, yet admitting of the conventional methods as particular cases. In fact, nonconservation laws, irreversibility, nonlocality, etc., admit as particular cases conservation laws, reversibility, locality, etc.

Problematic aspects of existing generalizations. Once the need for broader methods in theoretical biology is admitted, the main issue addressed in this monograph is their identification.

The main criteria per the selection of consistent generalized methods. After studying these problems for decades, I laboriously rediscovered a century old truth, that *there cannot be really new advances in sciences without really new mathematics.*

I attacked the problem when at Harvard University back in the early 1980's under support from the U. S. Debarment of Energy, by initiating the study of fundamentally novel mathematics, that is, I searched for *new numbers, new spaces, new algebras, new geometries,* etc., suitable for the objective at hand.

This monograph reports the outcome of my scientific journey which has apparently resulted in the identification of the desired new methods, with the understanding that I am a theoretical physicist, thus without the necessary training and knowledge in biology. My primary objective is therefore that of presenting the new *methods* themselves, with a mere indication of their potential in theoretical biology. The appraisal of the actual possibilities in biology can only be conducted by biologists in due time.

It may be of some value to indicate in these introductory words the main guiding principle used in the identification of the new methods.

Since the time of my graduate studies in theoretical physics at the University of Torino, Italy, in the late 1960's, I have been fascinated by the mathematical beauty, physical consistency and historical successes of quantum mechanics. Throughout this monograph I therefore assume quantum mechanics to be *exactly valid* under the conditions of its original conception and applicability of its mathematical structure, systems of particles and their interactions when they can be well approximated as being point-like under only action-at-a-distance/potential interactions (as we shall see, this can be reduced to mutual distances bigger than the coherent wavelength of each particle pair).

Nevertheless, since the time of my graduate studies I perceived clear imitations in such an exact applicability, evidently because nature cannot possibly be all reduced to isolated points, owing to the interactions due to wave-overlapping (at mutual distances smaller than the coherent wavelength of each pair) under which quantum mechanics i no longer exact on numerous topological, analytic, algebraic geometric and other grounds [c].

I therefore searched for generalized methods which : 1) preserve the main

irreversibility at the most ultimate possible level.

More recently, thanks to the patience by the mathematician T. Vougiouklis of the University of Thrace in Xanthi, Greece, in explaining the topic to me, I became aware that the *multivalued hyperstructures* can also be formulated in an axiom–preserving form, thus providing the most general known axiom–preserving liftings of conventional methods. The main novelty with respect to the genotopies is that the unit, elements and their operations are multi–valued, thus resulting to be particularly intriguing for the representation of nonconservative and irreversible biological systems with complex and locally differentiated internal structures.

In summary, this monograph outlines three levels of generalized methods I have tentatively called isotopies, genotopies and hyperstructures with progressive methodological characteristics for the representation of progressively more complex biological systems.

The important role of our sensory perception. In approaching preliminary applications of the new methods to biological systems I discovered that the main geometric principle had fundamental biological implications which should also be indicated in these introductory lines.

In essence, our sensory perception of biological structures have indeed limited capabilities, yet they are definitely capable of detecting deviations from conventional structures. The selection of generalized methods must therefore be compatible with our sensory perception.

As a concrete example, the visual inspection of a sea shell in our hands is based on our three Eustachian tubes which yield our *Euclidean perception* of the object considered. But, as we shall see, the Euclidean geometry is grossly insufficient for a consistent representation of the growth of sea shells, e.g., because it is structurally local–differential, conservative and reversible.

The problem therefore emerges of identifying a more adequate geometry for the quantitative representation of sea shells growth in an axiomatically consistent and invariant way.

At this point numerous possibilities emerge. First, one can attempts traditional generalizations, such as Riemannian, Finslerian, Desarguesian, etc. It is

the same methods still apply to both biological and physical worlds.

This leads in a natural way to doubt as to whether all differences between biological and physical systems are in reality so ponderable to admit a mathematical treatment at all.

I rediscovered in this way another century old truth, that *scientific disciplines will never admit final theories.*

Ruggero Maria Santilli
Baywood Village, Gulf of Mexico
Palm Harbor, Florida, July 1996

Permanent address:
Institute for Basic Research
P.O.Box 1577, Palm Harbor, FL 34682, U.S.A.
ibrrms@pinet.aip.org

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1: INTRODUCTION

1.1: Insufficiencies of contemporary mathematical methods for theoretical biology.

The mathematical methods currently used in theoretical biology (see, e.g., ref.s [1,2,3]) are essentially those of contemporary classical, quantum and statistical mechanics which are ultimately reducible to Lie symmetries and their underlying topologies, geometries and mechanics.

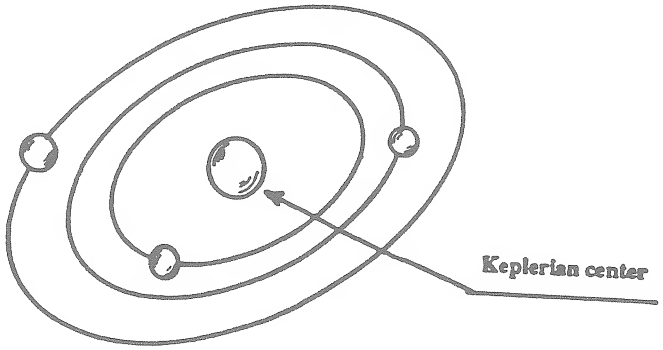
With the clear understanding that these methods have indeed produced outstanding results and are therefore valid for a study of theoretical biology, in this monograph we point out a number of insufficiencies which warrant the study of more general and more adequate methods.

The first, and perhaps most important insufficiency is of basic structural character. Contemporary mathematical methods have been conceived, developed, applied and verified for perennial and immutable conservative systems, such as a planetary or atomic system, in which they have achieved outstanding results of historical proportions.

On the contrary, biological structures are inherently nonconservative evidently because they grow in time, and then decay. It then follows that the fundamental laws of contemporary physics are not directly applicable to biological systems and, if applied without care, could lead to contradictions with reality. This is due to the fact that the entire body of contemporary methods have been built to represent *the stability of the orbits, conservation laws* and the like, while an effective representation of biological systems requires the

solar system. Galileo's conception of Jupiter as a massive point is therefore fully valid, thus providing solid physical foundations for the local–differential topology and mechanics.

PHYSICAL SYSTEMS



BIOLOGICAL SYSTEMS

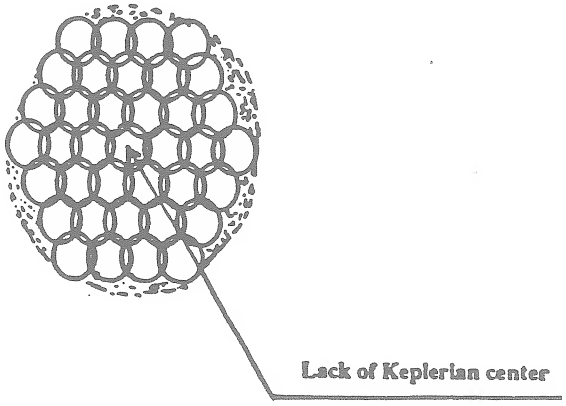


FIGURE 1.1. A schematic view illustrating the profound differences between physical and biological systems with consequential need for different methods. The mathematical methods of contemporary theoretical physics have been built for the

an excellent approximation of physical reality. In fact, their oblateness due to spin is very small, and their hyperdense character does indeed render them quite close to a perfectly rigid body.

The same effectiveness in particle physics constitutes clear insufficiencies when applied to theoretical biology. To begin, absolute and perfectly spherical shapes (that is, spherical shapes without protuberances) do not exist in the biological world. Moreover, the most evident characteristic emerging from a visual comparison, say, of a metal with a human limb is the rigidity of the former and the structural flexibility of the latter.

It then follows that mathematical methods which are effective for the characterization of the approximate rigidity in the physical world are structurally insufficient for a quantitative study of the manifest deformability of biological structures and, if applied without sufficient care, may lead to numerical results without connection with the biological reality.

Another significant difference between the physical and biological worlds is that the former is essentially dominated by *action–at–a–distance interactions derivable from a potential,* as it is typically the case of contemporary theoretical physics. In biological systems we have instead the dominance of the so–called *contact interactions,* i.e., interactions due to physical contact among atoms, molecules, etc. These interactions have *zero–range* by conception, and their representation with a potential has no biological or mathematical meaning.

As a result, physical systems are representable with a Lagrangian or a Hamiltonian within the context of their respective mechanics and their internal interactions are reducible to exxchanges of photons or other particles. On the contrary, biological systems are not entirely representable by a Lagrangian or a Hamiltonian owing to the indicated dominance of nonpotential over the potential interactions.[1]

[1] On technical grounds, contact interactions in three dimension are *variationally nonselfadjoint,* that is, they violate the necessary and sufficient conditions for the existence of a Lagrangian or a Hamiltonian, as studied in detail in monographs [4,5]. In certain elementary cases a Lagrangian or a Hamiltonian can be constructed under coordinate transformations to hypothetical reference frames. However, the latter are not realizable in laboratory, besides requiring a number of approximations (e.g., of local–

of isolated points, thus confirming again the need in biology of new methods, this time on topological grounds. A most dominant characteristic of the latter interactions is that, being also of contact type, *they occur without any exchange of energy*. In other words, contact interactions due to wave overlapping remain of zero-range in the sense that they simply cannot be mediated by particle exchanges. Nevertheless, their action is felt over a finite distance because their structure is defined over a finite volume. To put it differently, the search within purely physical settings of how a cell may influence another is not only bound to be fruitless, but also highly miopic and eventually misleading, for it is tacitly based on the complete reduction of biological structures to physical systems. The achievement of a quantitative representation of the novel nonlocal nonpotential interactions is a main objective of this monograph.

The need for nonlocal formulations is sufficient, alone, to require a profound revision of the contemporary mathematical methods. To begin, nonlocality requires its representation beginning at the level of a new topology. Second, nonlocality is inherently nonpotential because the addition of an hypothetical "nonlocal potential" to a Lagrangian or a Hamiltonian has no mathematical or physical sense. Finally, nonlocality is structurally nonhamiltonian, that is, it is beyond the representational capabilities of a Hamiltonian (we can technically say that the systems are *variationally nonselfadjoint,* i.e., they violate the integrability conditions for their sole representation via a first-order Lagrangian or a Hamiltonian [4,5]). The loss of a Lagrangian or Hamiltonian character implies the consequential inapplicability of the totality of the mathematical methods of contemporary physics, including classical and quantum mechanics, Lie's theory, conventional local-differential geometries, etc.

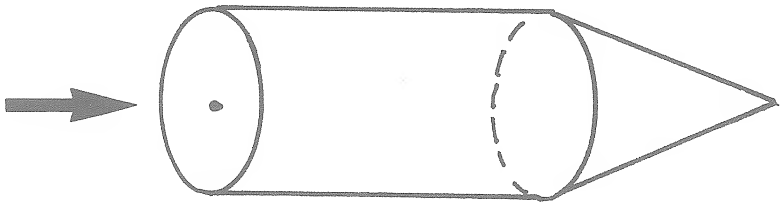
1.2: Basic requirements for generalized methods.

Without any need of uniqueness, in this monograph we shall explore certain generalizations of contemporary mathematical methods which have been selected, constructed and applied to verify the following conditions:

1] The new methods must admit conventional methods as particular

applications.

LOCOMOTION IN PHYSICS



LOCOMOTION IN BIOLOGY

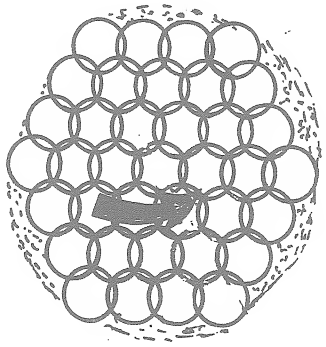


FIGURE 1.3: Another aspect in which fundamental differences are expected to emerge between physical and biological systems is that of *locomotion,* that is, the mechanisms according to which particles move. In physics motion occurs *only upon the application of a Newtonian force,* as typically the case of rockets. This is

Finally, readers should not reasonably expect that the new methods and their applications have achieved the same maturity as that of conventional methods [1,2,3]. In fact, the new mathematical methods and their application are at their first infancy, and so much remains to be done.

1.3: Outline of the new methods and guide to the primary literature.

The studies conducted by this author have indicated that *generalized methods verifying conditions 1)–5) of the preceding subsection are not unique, and actually constitute a chain of generalized methods of increasing complexity and methodological capabilities for the quantitative representation of biological structures of increasing complexity, which have been submitted under the names of*

$$\textit{Isotopies} \subset \textit{genotopies} \subset \textit{hyperstructures} \qquad (1.1.1)$$

This monograph is devoted to the presentation of the rudiments of the above methods and to an illustration of their applications to biological systems.

The *isotopies* were submitted by R. M. Santilli [6] when at the Department of Mathematics of Harvard University back in the late 1970's (see monographs [4,5] for a comprehensive presentation of these first studies and monographs [7–10] for more recent classical and operator treatments) and they are today called *Santilli's isotopies* in the independent literature in the field (see monographs [11–14] and quoted references).

The main idea of the isotopies from which the entire chain (1.1.1) can be uniquely derived is the generalization of the basic n×n–dimensional unit I = diag. (1, 1, 1,...) of conventional methods into an n×n matrix $\hat{1}$ which is well behaved, nonsingular and Hermitean, but whose elements otherwise possess an arbitrary,

in which case (only), $\hat{1}$ is called the *isounit* and $\hat{T}$ the isotopic element.

It was proved in the same original proposal [6] that an associative algebra ξ with unit I, elements A, B, ... and conventional associative product $A\times B$ is lifted by maps (1.1.2) and (1.1.3) in an axiom-preserving way into the new algebra $\hat{\xi}$ with new unit $\hat{1} = \hat{T}^{-1}$, and the new product $A\hat{\times}B = A\times\hat{T}\times B$. In fact, the new product is as associative as the original one

$$\xi:\ A\times(B\times C) = (A\times B)\times C \quad \to \hat{\xi}:\ A\hat{\times}(B\hat{\times}C) = (A\hat{\times}B)\hat{\times}C\,. \quad (1.1.6)$$

Moreover, it was proved in ref. [6] (via the lifting of the Poincaré-Birkhoff-Witt theorem of universal enveloping associative algebras) that the original and new algebra are locally isomorphic under the condition of positive-definiteness $\hat{1} > 0$, while they are antiautomorphic if $\hat{1} < 0$. The new structure $\hat{\xi}$ was then called *isoassociative algebra* to indicate the property that the associative law holds at the *isotopic* level.

Recall that the basic dynamical equations for the time evolution of conventional methods are characterized by the antisymmetric part ξ^- attached to the associative algebra ξ with familiar product $[A, B] = A\times B - B\times A$ which characterizes a *Lie algebra*, resulting in the time evolution $i\,dA/dt = [A, H]$, where H is the Hamiltonian, with exponentiated form $A(t) = \exp\{iHt\}\times A(0)\times\exp\{-itH\}$ constituting a one-parameter Lie group.

The isotopies permit a step-by-step structural generalizations of the above methodological lines. In fact, the antisymmetric algebra $\hat{\xi}^-$ attached to the isoassociative algebra $\hat{\xi}$ is given by

$$[A,\hat{B}] = A\hat{\times}B - B\hat{\times}A\,, \quad (1.1.7)$$

which was proved in ref. [6] to preserve the Lie axioms although at the isotopic level which characterizes an algebra today called *Lie-Santilli isoalgebra* [11-14].

differential calculus, trigonometric and hyperbolic functions, special functions and transforms, vector, metric and Hilbert spaces, algebras, geometries, mechanics, etc. [4–14]. As an example, the use of conventional numbers with unit +1 and generalized vector spaces with isounit $\hat{1}$ would imply profound inconsistencies. As we shall see, the generalizations are quite simple, yet unique and effective for the intended use.

The reader is therefore alerted that the inspection and appraisal of the content of this memoir via conventional mathematical methods leads to a number of inconsistencies which are generally undetected by non–experts in the field.

The second class of generalized methods, called *genotopies,* were also proposed by Santilli in the same memoirs [6] as a natural generalization of the isotopies when the isounit is no longer Hermitean,

$$\hat{1} \neq \hat{1}^{\dagger}, \tag{1.1.10}$$

This implies the *dual* lifting of the original unit I

$$I \rightarrow \hat{1}^{>} \quad \text{and} \quad I \rightarrow {}^{<}\hat{1}, \tag{1.1.11}$$

with an interconnecting conjugation

$$\hat{1}^{>} = ({}^{<}\hat{1})^{\dagger}. \tag{1.1.12}$$

as well as the *dual* lifting of the product

$$\xi : A \times B \rightarrow \hat{\xi}^{>} : A > B = A \times \hat{R} \times B,$$

$$\xi : A \times B \rightarrow {}^{<}\hat{\xi} : A < B = A \times \hat{S} \times B, \tag{1.1.13}$$

The above properties are trivial at the conventional and isotopic levels, but they acquire their full meaning at the broader genotopic level. In fact, The envelopes $\hat{\xi}^{\rightarrow}$ and $\hat{\xi}^{\leftarrow}$ are now replaced by the forward and backward structures $\hat{\xi}^{>}$ and $^{<}\hat{\xi}$, resulting in the characterization of the product first introduced in memoir [6b]

$$(A, B) = A < B - B > A = A \times \hat{R} \times B - B \times \hat{S} \times A, \tag{1.1.18}$$

which characterize a realization of the so-called *Lie-admissible algebras* proposed by Albert [20] back in 1948.

The fundamental dynamical equations of the genotopic theories, first proposed in memoir [6b], are then given by

$$i\, dA / dt = (A, H) = A < H - H > A = \tag{1.1.19}$$

$$= A \times \hat{R}(t, r, \dot{r}, \ddot{r}, \psi, \partial\psi, \partial\partial\psi, \tau, \mu, n, ...) \times H - H \times \hat{S}(t, r, \dot{r}, \ddot{r}, \psi, \partial\psi, \partial\partial\psi, \tau, \mu, n, ...) \times A$$

with exponentiated form

$$A(t) = (e^{iH \times \hat{R} \times t}) \times A(0) \times (e^{-it \times \hat{S} \times H}), \tag{1.1.20}$$

The above equations are the fundamental elements of the genotopies of Lie's theory, first proposed in memoir [6a], today called *Lie-Santilli genotheory,* and refer to the genotopies of enveloping algebras, Lie algebras, Lie groups, representation theory, etc.

As one can see by comparing the basic dynamical equations (1.1.8)-(1.1.9) and (1.1.19)-(1.1.20), *Santilli's genotopies are structurally irreversible,* that is, they are irreversible irrespective of the time behaviour of potentials, Lagrangians, Hamiltonians, generalized units, and all that. In fact, it is rather natural to associate the product ">" with motion forward in time, and the product "<" with motion backward in time, thus resulting in a representation of irreversibility

the original multiplication × is restricted *either* to be to the right *or* to the left, respectively, verify *individually* all axioms of a field (closure, distributive law, etc.).

This simple property persists under both isotopies and genotopies and permits the construction of new rigorous methods of genotopic type which we shall outline in this memoir and specialize to theoretical biology.

The above comments appear pure numerology at a first inspection but assume their fundamental role when applied to the fundamental content of all theories, their dynamical evolution. Let us reinspect together the time evolution currently used in all quantitative science, that characterized by the one-parameter Lie group of time evolution G(t): exp(iH×t)×A(0)×exp(−ith). It is evident that this structure is the result of *two* ordered actions, one to the right exp(iHt)×A(0) and one to the left A(0)×exp(−itH), and we shall write

$$G(t): \quad A(t) = e^{iH\times t} \rightarrow A(0) \leftarrow e^{-it\times H} \equiv e^{iH\times t} \times A(0) \times e^{-it\times H} . \qquad (1.1.21)$$

In fact, the above structure is mathematically called a *bimodule.*

In the transition to the isotopic law (1.20) we evidently p(reserve the above structure and only realize it in the more general isotopic way

$$\hat{G}(t): \quad A(t) = e^{iH\times\hat{T}\times t} \rightarrow A(0) \leftarrow e^{-it\times\hat{T}\times H} \equiv e^{iH\times\hat{T}\times t} \times A(0) \times e^{-it\times\hat{T}\times H} . \qquad (1.1.22)$$

The genotopies were submitted in memoir [6b] precisely on the basis of the above properties only generalized with a nontrivial difference of the action from the write and that to the left

$$<\hat{G}>(t): \quad A(t) = e^{iH\times\hat{S}\times t} \rightarrow A(0) \leftarrow e^{-it\times\hat{R}\times H} . \qquad (1.1.21)$$

Another important point needed to minimize possible misrepresentations is that *the genotopies require a dual generalization of the totality of mathematical methods, that is, a dual formulation of numbers, fields, angles, trigonometric functions, etc., one per each direction of time.*

$$\{ \hat{<1} \} = \{ \hat{<S} \}^{-1} : \qquad A \hat{<} \{ \hat{<1} \} \equiv A = \{ \hat{<1} \} \hat{<} A . \tag{1.1.25}$$

When the above properties are verified under *strong equalities* [16], the set $\{\hat{1>}\}$ ($\{\hat{<1}\}$) shall be called the *forward hyperunit (backward hyperunit),* and the set $\{\hat{R}\}$ ($\{\hat{S}\}$) shall be called the *forward hyperelement (backward hyperelement).*

The further broadening of the genotopies then lead to the fundamental; dynamical equations of the hyperstructures

$$i\, dA / dt = \{ A, H \} = A \hat{<} H - H \hat{>} A = \tag{1.1.26}$$

$$= A \hat{\times} \{ \hat{R}(t, r, \dot{r}, \ddot{r}, \psi, \partial\psi, \partial\partial\psi, \tau, \mu, n, ...) \} \hat{\times} H - H \hat{\times} \{ \hat{S}(t, r, \dot{r}, \ddot{r}, \psi, \partial\psi, \partial\partial\psi, \tau, \mu, n, ...) \} \hat{\times} A,$$

with exponentiated form

$$A(t) = (e^{iH \hat{\times} \{\hat{R}\} \hat{\times} t}) \times A(0) \times (e^{-it \hat{\times} \{\hat{S}\} \hat{\times} H}), \tag{1.1.27}$$

apparently introduced here for the first time.

The above equations indicate the existence of a step–by–step further generalization of Lie's theory, called *Lie–Santilli–Vougiouklis hypertheory,* and refer to the hypergeneralization of enveloping algebras, Lie algebras, Lie groups, representation theory, etc.

The most salient aspect emerging from the comparison of the above structure is that the product of two numbers, say 2 and 3, yields one single value for conventional numbers as well as for isonumbers and genonumbers, while *the product of two numbers yields a "set" of values for Santilli–Vougiouklis hyperstructures.*

This is precisely the *desired* feature because it is the most natural one for the representation of the growth in biological structures, e.g., the mathematical representation of one cell splitting into two which then split into four, etc., which can be represented via forward hyperunits consisting of a set of *two* elements $\{\hat{1>}\} = \{\hat{1>}_1, \hat{1>}_2\}$, the backward hyperunit not characterizing a real process, as *necessary* for irreversible growth in biology.

and quantum, analytic and geometric methods of isotopic, genotopic and hyperstructural type which can be constructed via the preceding basic notions. The applications considered in this memoir are primarily intended to point out that the geometric complexity of biological systems is dramatically beyond the simplistic capabilities of our limited senses.

The analytic and geometric methods presented in this memoir should be complemented with a study of the algebraic–group theoretical parts. In fact, the totality of the applications introduced in this memoir are reducible to primitive generalized symmetries. For comprehensive study of the Lie–Santilli isotheory we refer the interested reader to the monograph by Sourlas and Tsagas [13] or to the readable review by Kadeisvili [21]. An outline of the fundamental isotopies of the rotational symmetry is presented in App. C for the reader's convenience. No review exists at this writing on the more general Lie–Santilli genotheory or on the still broader Lie–Santilli–Vougiouklis hypertheory.

On historical grounds we should mention that an extensive search conducted by this author in conjunction of the writing of memoir [6a]. The search indicated that the notion of isotopy is rather old. As Bruck [22] recalls, the notion can be traced back to the early stages of set theory where two Latin squares were said to be *isotopically related* when they can be made to coincide via permutations. Since Latin square can be interpreted as the multiplication table of quasigroups, the isotopies propagated to quasigroups and then to Jordan algebras (see, e.g., McCrimmon [23]). Studies on the isotopies of the unit; fields, vector spaces, Lie's theory and other methods were initiated in [6a]. An exhaustive literature on isotopies up to 1984 can be found in bibliography [24] while subsequent references can be found in the recent monograph by Lõhmus, Paal and Sorgsepp [14]. As indicated earlier, no prior contributions appear to exist, to this author best knowledge, on genotopies prior to memoir [6a] and on the hyperstructures with a unit prior to ref. [16].

While inspecting this memoir, the reader should finally keep in mind that this author is a theoretical *physicist* and not a theoretical *biologist*. The primary objective of this memoir is that of presenting *generalized methods* with a mere *indication* of their possibilities. The study of rigorous, extensive and in depth applications to theoretical *biology* is the task of theoretical *biologists*.

theories are indistinguishable at the abstract, realization–free level by conception and construction.

Once the simplest possible unit +1 is abandoned in favor of a more general notion, a considerable richness of the isotopic structure emerges because of various different characteristics of the isounit $\hat{1}$ which are evidently absent for the unit +1.

The different structural characteristics of the isounit evidently carry over to the entire isotopic theory. These different possibilities were classified for the first time by Kadeisvili [25] resulting in what is today called the *Kadeisvili classification of isotopies* according to the following classes:

Class I, occurring for isounits which are well behaved, nowhere singular, Hermitean and positive definite, $\hat{1} > 0$;

Class II: occurring for isounits which are well behaved, nowhere singular, Hermitean and negative definite, $\hat{1} < 0$;

Class III, which is the union of Classes I and II, in which case the isounit has an undefined signature;

Class IV, which is the union of Classes I, II, III plus the admission of singular isounits, $\hat{1} = 0$;

Class V, which is the union of all preceding classes, plus the possibility that the isounit has arbitrary characteristics, and can therefore be a discrete quantity, a distribution, a step function, a lattice, etc.

As established in monographs [9–12], the isotopies of Class I ($\hat{1} > 0$) have permitted a novel representation of matter, while those of Class II ($\hat{1}<0$) have produced a novel representation of antimatter. No known application of the isotopies of Class III (Isounits with undefined signature) has been identified in physics until now.

The task of this monograph begins with the indication that *isotopies particularly significant for theoretical biology are those of Classes III and IV.* As we shall see in the applications, this is *necessary* for a quantitative representation of the bifurcations and numerous other cases.

As we shall see, the isotopies of Class III imply the possibility of continuously moving from motion forward to motion backward in time. We can

$$\hat{1} \hat{\times} \hat{a} = \hat{a} \hat{\times} \hat{1} \equiv \hat{a}, \ \forall \ \hat{a} \in \hat{F}, \qquad (2.1.3)$$

called "isounit". Under these assumptions $\hat{F}$ *is a field, i.e., it satisfies all properties of* F *in their isotopic form:*

1. *The set* $\hat{F}$ *is closed under addition,* $\hat{a} + \hat{b} \in \hat{F}, \forall \ \hat{a}, \hat{b} \in \hat{F}$.
2. *The addition is commutative,* $\hat{a} + \hat{b} = \hat{b} + \hat{a}, \forall \hat{a}, \hat{b} \in \hat{F}$,
3. *The addition is associative,* $\hat{a} + (\hat{b} + \hat{c}) = (\hat{a} + \hat{b}) + \hat{c}, \forall \hat{a}, \hat{b}, \hat{c}, \in F$,
4. *There is an element* 0, *called t'additive unit", such that* $\hat{a} + 0 = 0 + \hat{a} = \hat{a}, \forall \hat{a} \in \hat{F}$
5. *For each element* $\hat{a} \in \hat{F}$, *there is an element* $-\hat{a} \in \hat{F}$, *called the "opposite of* $\hat{a}$", *which is such that* $\hat{a} + (-\hat{a}) = 0$;
6. *The set* $\hat{F}$ *is closed under isomultiplication,* $\hat{a} \hat{\times} \hat{b} \in \hat{F}, \forall \hat{a}, \hat{b} \in \hat{F}$,
7. *The multiplication is generally non–isocommutive,* $\hat{a} \hat{\times} \hat{b} \neq \hat{b} \hat{\times} \hat{a}$, *but "isoassociative",* $\hat{a} \hat{\times} (\hat{b} \hat{\times} \hat{c}) = (\hat{a} \hat{\times} \hat{b}) \hat{\times} \hat{c}, \forall \hat{a}, \hat{b}, \hat{c} \in \hat{F}$;
8. *The quantity* $\hat{1}$ *in the factorization* $\hat{a} = a \hat{\times} \hat{1}$ *is the "multiplicative isounit" of* $\hat{F}$ *as per Eq.s* (1.3)
9. *For each element* $\hat{a} \in \hat{F}$, *there is an element* $\hat{a}^{-\hat{1}} \in \hat{F}$, *called the "isoinverse", which is such that* $\hat{a} \hat{\times} (\hat{a}^{-\hat{1}}) = (\hat{a}^{-\hat{1}}) \hat{\times} \hat{a} = \hat{1}$.
10. *The set* $\hat{F}$ *is closed under joint isomultiplication and addition,*

$$\hat{a} \hat{\times} (\hat{b} + \hat{c}) \in \hat{F}, \quad (\hat{a} + \hat{b}) \hat{\times} \hat{c} \in \hat{F}, \ \forall \ \hat{a}, \hat{b}, \hat{c} \in \hat{F}; \qquad (2.1.4)$$

11. *All elements* $\hat{a}, \hat{b}, \hat{c} \in \hat{F}$ *verify the right and left "isodistributive laws"*

$$\hat{a} \hat{\times} (\hat{b} + \hat{c}) = \hat{a} \hat{\times} \hat{b} + \hat{a} \hat{\times} \hat{c}, (\hat{a} + \hat{b}) \hat{\times} \hat{c} = \hat{a} \hat{\times} \hat{c} + \hat{b} \hat{\times} \hat{c}, \qquad (2.1.5)$$

When there exists a least positive isointeger $\hat{p}$ *such that the equation* $\hat{p} \hat{\times} \hat{a} = 0$ *admits solution for all elements* $\hat{a} \in \hat{F}$, *then* $\hat{F}$ *is said to have "isocharacteristic* p'. *Otherwise,* $\hat{F}$ *is said to have "isocharacteristic zero". Unless otherwise stated, all isofields considered hereon shall be Class III isofields of isocharacteristic zero.*

$$\hat{F}_{III}(\hat{a},+,\hat{x}) = \{ \hat{F}_{I}(\hat{a},+,\hat{x}), \hat{1} > 0, \hat{F}_{II}(\hat{a},+,\hat{x}), \hat{1} < 0 \}, \tag{2.1.6}$$

with interconnecting map called *isoduality*

$$\hat{1} > 0 \ \rightarrow \ \hat{1}^{d} = -\hat{1} < 0, \tag{2.1.7}$$

and introduced for the first time by this author in ref.s [27]. The Class II isofields are also written in the literature $\hat{F}^{d}(\hat{a}^{d},+,\hat{x}^{d})$ and called *isodual isofields* with *isodual isonumbers* $\hat{a}^{d} = a\times\hat{1}^{d} = -\hat{a}$, and isodual isoproduct $\hat{x}^{d} = \times\hat{1}^{d}\times = -\hat{x}$.

Since the two branches are topologically disjoint, they can be treated separately. Their continuous union can be studied at the level of Class IV with the inclusion of the value $\hat{1} = 0$ which is omitted for brevity.

The *isonorm* of an isofield is defined by

$$\hat{|a|} = |a|\times\hat{1}, \tag{2.1.8}$$

where $|a|$ is the conventional norm.

It is therefore easy to see that *the isonorm of isofields of Class I is positive-definite, while that of Class II is negative definite.* This implies that all physical, chemical or biological characteristics which are conventionally positive, becomes negative-definite when lifted into isodual isofields.

One should keep in mind the complete equivalence of positive-definite characteristics referred a positive-definite unit and negative-definite characteristics referred to a negative-definite unit. This seemingly irrelevant property of the theory of isonumbers has rather profound implications, inasmuch as it implies the full causality of motion backward in time when referred to a negative time unit, in view of its complete equivalence of our ordinary perception of motion forward in time which is *tacitly* referred to a positive-definite unit.

The isotopies therefore permit the discovery of entire new classes of numbers and their isoduals, which admit as particular case the conventional

The next simplest possible example is the representation of systems which are open–nonconservative because of exchanges of physical quantities with an external system. In this case the isounit is a well behaved function of local quantities admitting of the value I as a particular case, e.g.,

$$\hat{1} = e^{f(t, x, \dot{x}, ...)}, \quad \hat{1}_{|f=0} = I . \tag{2.1.13}$$

Isounits of this type permit the representation of continuously decaying angular momenta; particles moving within resistive media under nonhamiltonian but local–differential forces (see later on); the growth of sea shells; and other nonconservative systems.

The next class of isounits used in applications is of nonlocal–integral type, that is, dependent on an integral over a surface or a volume. An illustration is given by the two electrons of the Cooper pair in superconductivity which experience an *attractive* interaction against their *repulsive* Coulomb force. The use of the quantum mechanical Coulomb law with conventional unit I = diag. (1, 1, 1) leads to repulsion. On the contrary, as studied in the applications, lifting of quantum mechanics via the use of the following isounit, called *animalu isounit,*

$$\hat{1} = e^{\int d^3x \, \phi^\dagger_\uparrow(x) \, \phi_\downarrow(x)} \text{diag. } (1, 1, 1) , \tag{2.1.14}$$

permits a quantitative interpretation of the attraction among the two identical electrons in a way which conforms with experimental evidence, where $\phi_\uparrow$ and $\phi_\downarrow$ are the wavefunctions of the two electrons with related spin orientation $\uparrow$ and $\downarrow$. The exponent then illustrates the type of nonlocality which we have been referring to. Note that when the overlapping of the two wavepackets $\phi_\uparrow$ and $\phi_\downarrow$ is no longer appreciable, the integral in the exponent of the isounit is null and $\hat{1}$ recovers the conventional unit I.

Note also that the notion of action–at–a–distance potential and related energy has no meaning of any nature for the interactions due to the *physical contact and overlap* of wavepackets. After all, these interactions are *zero–range by conception.* As such, contact interactions should be represented in general

where δ = diag. (1, 1, 1) *is the conventional Euclidean metric, local chart in contravariant and covariant forms*

$$\hat{x} = \{ \hat{x}^k \} = \{ x^k \times \hat{1} \}, \quad \hat{x}_k = \hat{\delta}_{ki} \hat{x}^i = \hat{T}_k{}^r \delta_{ri} x^i \times \hat{1}, \quad x^k, x_k \in E ; \qquad (2.1.17)$$

and "isoseparation" among two points $\hat{x}, \hat{y} \in \hat{E}$

$$(\hat{x} - \hat{y})^{\hat{2}} = [(\hat{x} - \hat{y}^i) \times \hat{\delta}_{ij} \times (\hat{x}^j - \hat{y}^j)] \times \hat{1} \in \hat{F} . \qquad (2.1.18)$$

The "isoeuclidean geometry" is the geometry of the isoeuclidean spaces.

The primary property of the lifting $E(x,\delta,R) \to \hat{E}(\hat{x},\hat{\delta},\hat{R})$ is the preservation of the original geometric axioms, thus characterizing an isotopy. In actuality, $E(x,\delta,R)$ and $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ coincide at the abstract level by construction for all positive-definite isounits $\hat{1}$ (but not so for isounits of different topology [11]). This is due to the construction of the isospaces via the deformation of the metric δ into the isometric $\hat{\delta} = \hat{T} \times \delta$ while jointly the original unit I is deformed in the amount *inverse* of the deformation of δ, $\hat{1} = \hat{T}^{-1}$. This mechanism then ensures the preservation of all original geometric properties, as studied later on.

Note that the isoseparation $\hat{x}^2$, for consistency, must be an element of the isofield, thus being isoreal number, that is, must have the structure of a number n multiplied by the isounit $\hat{1}$. This isoscalar character is expressed by the isomultiplication

$$\hat{x}^2 = \hat{x}^k \hat{\times} \hat{x}_k = (x^k \times \hat{1}) \times \hat{T} \times (x_k \times \hat{1}) = (x^k \times x_k) \times \hat{1} = n \times \hat{1} . \qquad (2.1.19)$$

But the contraction over the repeated index k is in isospace, we recover in this way the isoseparation of Def. 2.2,

$$\hat{x}^2 = (x^k \times x_k) \times \hat{1} = (x^i \times \hat{\delta}_{ij} \times x^j) \times \hat{1} . \qquad (2.1.20)$$

have a curvature much broader than that of the Riemannian spaces.

It is easy to see that

Lemma 2.1: *Isoeuclidean and isominkowskian spaces of Class III with the same dimension coincide.*

Proof. The conventional (3+1)–dimensional minkowskian space with metric η = diag. (1, 1, 1, –1) is a particular case of the 4–dimensional isoeuclidean space of Class III in which the identity $\hat{\delta} \equiv \eta$ is possible due to the indefinite signature of $\hat{1}$, and the same result persists under isotopy. **q.e.d.**

We begin to address in this way the conceptual complexities of the isospaces and their characteristics of being beyond our routine intuition. In fact, Einstein's notion of equivalence between space and time is much enlarged to include arbitrarily evolution forward and backward in time in a symbiotic unification of Euclidean, Minkowskian and Riemannian spaces.

The reader can now begin to see the reason why the primary objective of this memoir is to study the application to theoretical biology of the isogeometries because, once the geometrical profiles appear meaningful and promising, the connected analytic and algebraic profiles are mere consequences.

2.1.D: Kadeisvili isocontinuity and Tsagas-Sourlas isotopology

The notion of *isocontinuity* on an isospace was first studied by Kadeisvili [25] and was shown to be easily reducible to that of conventional continuity because the *isomodulus* $\hat{|}\,\hat{f}(\hat{x})\,\hat{|}$ of a function $\hat{f}(\hat{x})$ on $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ over $\hat{R}(\hat{n},\hat{+},\hat{\times})$ is given by the conventional modulus $|\hat{f}(\hat{x})|$ multiplied by the positive–definite isounit $\hat{1}$,

$$\hat{|}\,\hat{f}(\hat{x})\,\hat{|} = |\hat{f}(\hat{x})|\times\hat{1} > 0\,. \qquad (2.1.21)$$

As an illustration, an infinite sequence $\hat{f}_1$, $\hat{f}_2$, ... is said to be *strongly*

topology on R^N,

$$\hat{\tau} = \{ \emptyset , \hat{R}^N, \hat{B}_i \} , \tag{2.1.26}$$

where $\hat{B}_i$ represents the subset of $\hat{R}^n$ defined by

$$\hat{B}_i = \{ \hat{P} = (\hat{a}_1, \hat{a}_2, ..., \hat{a}_n) \ni \hat{n}_i < \hat{a}_1, \hat{a}_2, ..., \hat{a}_n < \hat{m}_i, \hat{n}_i , \hat{m}_i , a_k \in \hat{R} \} . \tag{2.27}$$

As one can see, the above topology coincides everywhere with the conventional topology τ of R^n *except at the isounit* $\hat{1}$. In particular, $\hat{\tau}$ is everywhere local-differential, except at $\hat{1}$ which can incorporate integral terms. The above structure is then called the *Tsagas-Sourlas isotopology* or an *integro-differential topology.*

Definition 2.3 [30]: *A "topological isospace"* $\hat{\tau}(\hat{R}^N)$ *is the isospace of Class III* $\hat{R}^N$ *equipped with the isotopology* $\hat{\tau}$. *A "Cartesian isomanifold"* $\hat{M}(\hat{R}^N)$ *is the topological isospace* $\hat{\tau}(\hat{R}^N)$ *equipped with a vector structure, an affine structure and the mapping*

$$\hat{1} : \hat{R}^n \to \hat{R}^n , \quad \hat{1} : \hat{a} \to \hat{1}(\hat{a}) = \hat{a}, \quad \forall \ \hat{a} \in \hat{R} . \tag{2.1.28}$$

An "isoeuclidean isomanifold" $\hat{M}(\hat{E}(\hat{x},\hat{\delta},\hat{R}))$ *occurs when the* N*-dimensional isospace* $\hat{E}$ *is realized as the Cartesian product*

$$\hat{E}(x,\hat{\delta},\hat{R}) \approx \hat{R}_1 \times \hat{R}_2 \times \ ... \ \times \hat{R}_N , \tag{2.1.29}$$

and equipped with the isotopology $\hat{\tau}$ *with isounit* (1.15).

The extension of the above definition to nondiagonal isounits $\hat{1}$ can be trivially achieved, e.g., by assuming that the individual isounits $\hat{1}_k$ are positive-definite N×N-dimensional nondiagonal matrices such to yield the assumed total unit $\hat{1}$ via the ordered Cartesian product

$$x^i \delta_{ij} x^j = x_i \delta^{ij} x_j = x^i x_j = x_i x^j, \quad \delta^{ij} = [(\delta_{mn})^{-1}]^{ij}. \tag{2.1.31}$$

Let $\hat{M}[\hat{E}(\hat{x},\hat{\delta},\hat{R})]$ be an isomanifold on $\hat{E}$ as per Definition 3 hereon referred as $\hat{M}(\hat{E})$. The *isodifferential calculus* on $\hat{M}(\hat{E})$ was identified for the first time by Santilli in the first edition of monograph [12] of 1994 and studied in more details in the recent papers [21–35] as an isotopic lifting of the conventional differential calculus on M(E), that is, a lifting based on the generalization of the unit I of M(E) into the isounit $\hat{I}$ of $\hat{M}(\hat{E})$, under the condition of preserving the axioms and properties of the ordinary differential calculus, including the condition of the invariance of the isounit (see below).

Definition 2.4: *The "first-order isodifferentials" of the contravariant and covariant coordinates* $\hat{x}^k$ *and* $\hat{x}_k$, *on* $\hat{M}(\hat{E})$ *are given by*

$$\hat{d}\, \hat{x}^k = \hat{I}_k{}^i(x, ...)\, dx^i, \quad \hat{d}\, \hat{x}_k = \hat{T}_k{}^i(x, ...)\, dx_i, \tag{2.1.32}$$

where the expressions $\hat{d}\hat{x}^k$ *and* $\hat{d}\hat{x}_k$ *are defined on* $\hat{M}(\hat{E})$ *while the corresponding expressions* $\hat{I}^k{}_i dx^i$ *and* $\hat{T}_k{}^i dx_i$ *are the projections on* M(E). *Let* $\hat{f}(\hat{x})$ *be a sufficiently smooth isofunction on a closed domain* $\hat{D}(\hat{x}^k)$ *of contravariant coordinates* $\hat{x}^k$ *on* M(Ê). *Then the "isoderivative" at a point* $\hat{a}^k \in \hat{D}(\hat{x}^k)$ *is given by*

$$\hat{f}'(\hat{a}^k) = \frac{\hat{\partial}\, \hat{f}(\hat{x})}{\hat{\partial}\, \hat{x}^k} \Big|_{\hat{x}^k = \hat{a}^k} = \hat{T}_k{}^i \frac{\partial f(x)}{\partial x^i} \Big|_{\hat{x}^k = \hat{a}} = \mathrm{Lim}_{\hat{d}\hat{x}^k \to \hat{0}^k} \frac{\hat{f}(\hat{a}^k + \hat{d}\hat{x}^k) - \hat{f}(\hat{a}^k)}{\hat{d}x^k} \tag{2.1.33}$$

where we assume Kadeisvili's [25] *notions of isocontinuity, isolimits and isoconvergence,* $\hat{\partial}\hat{f}(\hat{x})/\hat{\partial}\hat{x}^k$ *is computed on* $\hat{M}(\hat{E})$ *and* $\hat{T}_k{}^i \partial f(x)/\partial x^i$ *is the projection in* M(E). *The "isoderivative" of a smooth isofunction* $\hat{f}(\hat{x})$ *of the covariant variable* $\hat{x}_k$ *at the point* $\hat{a}_k \in \hat{D}(\hat{x}_k)$ *is given by*

$$\hat{f}'(\hat{a}_k) = \frac{\hat{\partial}\, \hat{f}(\hat{x})}{\hat{\partial}\, \hat{x}_k} \Big|_{\hat{x}_k = \hat{a}_k} = \hat{I}_k{}^i \frac{\partial f(x)}{\partial x_i} \Big|_{\hat{x}_k = \hat{a}_k} = \mathrm{Lim}_{\hat{d}\hat{x}_k \to \hat{0}_k} \frac{\hat{f}(\hat{a}_k + \hat{d}x_k) - \hat{f}(\hat{a}_k)}{\hat{d}\hat{x}_k}$$

$$\frac{\hat{\partial}\,(\hat{x}_k \hat{x}^K)}{\hat{\partial}\hat{x}^r} = \frac{\hat{\partial}\,(\hat{x}^I \hat{\delta}_{ij} \hat{x}^J)}{\partial x^r} = \hat{T}_r{}^i \frac{\partial\,(x^I \delta_{ij} x^J)}{\partial x^i} = \hat{T}_r{}^i\, 2\, x^i = 2\, \hat{x}_r\,,$$

$$\frac{\hat{\partial}\,\ln \hat{\psi}(\hat{x})}{\hat{\partial}\hat{x}^k} = \hat{T}_k{}^i \frac{\partial\,\ln \psi(x)}{\partial x^i} = \frac{1}{\hat{\psi}(\hat{x})} \frac{\hat{\partial}\,\hat{\psi}(\hat{x})}{\hat{\partial}\,\hat{x}^k}\,, \qquad (2.1.39)$$

and similarly for other cases.

For completeness we mention the (indefinite) *isointegration* which, when defined as the inverse of the isodifferential, is given by

$$\hat{\int} \hat{d}\hat{x} = \int \hat{T}\hat{1}\, dx = \int dx = x\,, \qquad (2.1.40)$$

namely, $\hat{\int} = \int \hat{T}$. Definite isointegrals are formulated accordingly.

The above basic notions are sufficient for our needs at this time. The class of isodifferentiable isofunctions of order m will be indicated $\hat{C}^m$.

An important property is that the isodifferential, isoderivative and isodifferentiation verify the condition of preserving the basic isounit $\hat{1}$. Mathematically, this condition is *necessary* to prevent that a set of isofunctions $\hat{f}(\hat{x})$, $\hat{g}(\hat{x})$, ..., on $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ over the isofield $\hat{R}(\hat{n},+,\hat{x})$ with isounit $\hat{1}$ is mapped under isoderivative into a set of isofunctions $\hat{f}'(\hat{x})$, $\hat{g}'(\hat{x})$, ..., defined over a *different* field because of the alteration of the isounit. Physically, the condition is also *necessary* because the unit is a pre-requisite for measurements. The lack of conservation of the unit therefore implies the lack of consistent physical applications.

As an example, the following alternative definition of the isodifferential

$$\hat{d}\hat{x}^k = d\,(\hat{1}^k{}_i\, x^i) = [\,(\partial_i \hat{1}^k{}_r)\, x^r + \hat{1}^k{}_i\,]\, dx^i = \hat{W}^k{}_i\, dx^i\,, \qquad (2.1.41)$$

would imply the alteration of the isounit, $\hat{1} \to \hat{W} \neq \hat{1}$, thus being mathematically and physically unacceptable.

Nevertheless, when using isoderivatives on independent isomanifolds, say, isoderivatives on coordinates and time, the above rule does not apply and we

geometry, such as the definition of angles, the notion of straight, perpendicular and parallel lines, etc. At the same time, the isometric possesses the most general possible functional dependence, $\hat{\delta} = \hat{\delta}(t, r, \dot{r}, \ddot{r})$, thus including as particular cases the Riemannian, Finslerian, Labacevskiian, nondesarguesian, or any other possible *noneuclidean* geometry in the same dimension.

The ultimate meaning of the n-dimensional isoeuclidean geometry of Class I which will emerge from our studies is that of unifying all possible geometries with the same dimension and signature. The isoeuclidean geometry of Class III unifies all possible geometries of the same dimension irrespective of their signature. Finally, the isogeometries of Classes IV and V are basically novel and vastly unexplored at this writing.

The reader should finally recall that, while conventional geometries have a unique formulation, isogeometries have a *dual* formulation, the first in isospace over isofields and the second via the projection in the original space over conventional fields. The same dual character persists for all possible isotopies.

2.2.B: Basic properties of isoeuclidean geometry.

Let us begin by studying first the axiom-preserving content of the isoeuclidean geometry, with particular attention to the image under isotopies of flatness, while curvature and other noneuclidean aspects will be considered later on.

By conception and construction, the reader should expect no deviation from the abstract axioms of the Euclidean geometry *under the conditions that the isounit is positive-definite and the isogeometry is computed in isospace over isofields.*

However, when the isoeuclidean geometry is projected in the conventional Euclidean space, new geometric features are expected to occur and the same is the case when the basic unit can be of Class greater than I.

As we shall see, the results are the same irrespective of whether one considers the abstract approach by Euclide and Hilbert or the coordinate

This seemingly innocuous occurrence has a number of intriguing mathematical implications and physical applications, such as it permits the *mathematical* conception of a new propulsion called *geometric propulsion,* introduced apparently for the first time in ref. [11] which, as we shall see better later on, is essentially based on *the motion of a point from one isocoordinates to another via the alteration of the underlying geometry, rather than the actual displacement of the point itself.* As we shall see, this property may have intriguing and far reaching implications in theoretical biology.

The study of the remaining properties of an isostraight line is left to the interested reader.

We now introduce the isotopies of the three–dimensional Euclidean vector space $V(r,+,\odot,R(n,+,\times)$.[2] We introduce in V a system of *Cartesian coordinates*, name a system in which *all axes have the same (dimensionless) unit* +1 and are perpendicular to each other.[3] In this way, the Euclidean vector admit the familiar components along the three axes $r = (x, y, z)$.

We shall continue to use our main notation whereby quantities with the "hat" are computed in isospace and quantities without are computed in their projection in the original space. The symbol $+ \equiv \hat{+}$ will be used without a "hat" under isotopies to recall the fundamental assumption of Ch. I.2 that the lifting of the sum implies the divergence of the exponentiation and other undesirable features.

Definition 2.6: *The isotopies of Class I of the three–dimensional Euclidean vector space* $V(r,+,\odot,R(n,+,\times))$, $r = (x, y, z)$, *called the three–dimensional "isoeuclidean isovector space", are given by the same original set of contravariant vectors reformulated as "isovectors"* $\hat{r} = r\times\hat{I} = (\hat{x}, \hat{y}, \hat{z}) = (x\times\hat{I},$

[2] Note the different products in a vector space, the product $\times \in R$ for numbers and the product $\odot \in V$ for vectors. Such a difference evidently persists under isotopies.

[3] Note that *noncartesian coordinate systems* also exist in the literature in which different axes have different units, but they are all referred to the *same field* and related basic unit. By comparison, the *isocartesian coordinates systems* have different units for different axes whose tensorial product is assumed as the basic unit of the underlying field. As a result, *noncartesian and isocartesian coordinate systems are inequivalent.*

shall see, this dual lifting permits the preservation of all original axioms. The basic quantity of the Euclidean geometry which remains invariant under lifting is therefore the quantity:

$$\text{Length} \times \text{Unit} \equiv \text{Isolength} \times \text{Isounit} . \qquad (2.2.4)$$

The realization of the isoeuclidean spaces of Class I primarily studied until now (spring 1996) is that characterized by diagonal isotopic elements and isounits. The proof that Definition 2.6. permits the preservation of the Euclidean axioms therefore exists only for the above particular form which is assumed hereon.

We therefore study the three-dimensional *isoeuclidean geometry* on the isospace of the same dimension with diagonal Class I isotopic elements and isounits, which can be written

$$\hat{E}(\hat{r},\hat{\delta},\hat{R}): \ \hat{r} = \{ \hat{r}^k \} \equiv \{ r^k \times \hat{1} \}, \quad \hat{r}_k = \hat{\delta}_{ki} \hat{r}^i \times \hat{1} \neq r_k \times \hat{1} ,$$

$$\hat{\delta} = \hat{\delta}^\dagger = \hat{T} \times \delta = (\hat{T}_i^{\ k} \times \delta_{kj}) = (\hat{\delta}_{ij}), \quad \delta = \text{diag.}(1, 1, 1),$$

$$\hat{T} = \hat{T}(t, r, \dot{r}, \ddot{r}, ...) = \text{diag.}(b_1^{\,2}, b_2^{\,2}, b_3^{\,2}) = \hat{T}^\dagger > 0 , \ b_k > 0 ,$$

$$\hat{1} = \hat{T}^{-1} = \text{diag.}(b_1^{\,-2}, b_2^{\,-2}, b_3^{\,-2}), \ \hat{\delta}^{ij} = \hat{1}^i_{\ k} \times \delta^{kj} , \ \hat{\delta}_{ik} \times \hat{\delta}^{kj} = \delta_i^{\ j} ,$$

$$\hat{r}^{\hat{2}} = (r^i \hat{\delta}_{ij} r^j) \times \hat{1} =$$

$$(x\, b_1^{\,2}\, x + y\, b_2^{\,2}\, y + x\, b_3^{\,2}\, z) \times \hat{1} \in \hat{R}(\hat{n},+,*) , \ i, j, k = 1, 2, 3. \qquad (2.2.5)$$

The most important differences between the Euclidean and isoeuclidean spaces are the following. The Euclidean space has the single and unique basic unit I = diag. (1, 1) which is the unit of the SO(3) symmetry, and which essentially implies *the same dimensionless unit* +1 *for all axes,*

$$\hat{r}^{\hat{2}} = (r_k \times r^k) \times \hat{1} = (r^i \times \hat{T} \times \delta_{ij} \times r^j) \times \hat{1} = r^2 \times \hat{T} \times \hat{T}^{-1} \equiv r^2 . \qquad (2.2.10)$$

This simple property illustrates the "hidden" character of geometric isotopies, and provides a reason why they have remained undetected for centuries until recently. Moreover, it is evident that the "hidden" character persists for arbitrary isotopic elements, because of the identity of the conventional and isotopic separation at the abstract level.

The 'hidden" character of the isoeuclidean geometry is also at the foundations of a number of applications to theoretical biology. In fact, it establishes that these applications are as geometrically sound as the use of the conventional geometry, thus preferring the isogeometry over the conventional one for its structurally broader possibilities.

Note also that the possible assumption of the basic invariant Length/Unit, rather than Length×Unit, would imply a geometry different than the isogeometry because characterized by the liftings $\delta \to \hat{\delta} = \hat{1} \times \delta$ and $I \to \hat{1}$. In this case the liftings are no longer "hidden" because property (2.52) no longer holds.

The *isodistance* between two points $\hat{P}_1(\hat{x}_1, \hat{y}_1, \hat{z}_1)$ and $\hat{P}_2(\hat{x}_2, \hat{y}_2, \hat{z}_2)$ of the isoeuclidean geometry is the isoscalar

$$\hat{D}_{12} = \hat{|} (\hat{r}_1 - \hat{r}_2) \hat{|} =$$

$$= [(x_1 - x_2)^2 b_1^2 + (y_1 - y_2)^2 b_2^2 + (z_1 - z_2) b_3^2]^{1/2} \times \hat{1} \in \hat{R} , \qquad (2.2.11)$$

where $\hat{r}_1$ and $\hat{r}_2$ are the isovectors from the origin to $\hat{P}_1$ and $\hat{P}_2$, respectively.

A primary implication of the notion of isodistance is that of *altering* the conventional Euclidean distance among two points according to the following:

Proposition 2.1: *Let* d_{12} *be the conventional Euclidean distance between two points* $P_1(x_1, y_1, z_1)$ *and* $P_2(x_2, y_2, z_2)$, *and let* $\hat{D}_{12} = D_{12} \times \hat{1}$ *be the corresponding isoeuclidean distance among two isopoints* $\hat{P}_1(\hat{x}_1, \hat{y}_1, \hat{z}_1)$ *and* $\hat{P}_2(\hat{x}_2, \hat{y}_2, \hat{z}_2)$, $\hat{x}_k = x_k \times \hat{1}$, $\hat{y}_k = y_k \times \hat{1}$, $\hat{z}_k = z_k \times \hat{1}$ *with the same coordinates* x_k , y_k, z_k, $k = 1, 2$, *of the*

shape and volume varying in time. In fact, the *same* k-side has *different* values depending on whether referred to the unit $I_k = +1$ or $\hat{1}_k = b_k^{-2}$, resulting in a volume for the interior observer which is arbitrarily smaller or bigger then that of the external observer. Also, equal sides for the outside observer are generally different among themselves when referred to *different units for different axes,* $\hat{1}_x \neq \hat{1}_y \neq \hat{1}_z \neq +1$, resulting in different shapes. Finally, the length of the sides is constant in time for the outside observer, while it may vary in time for the interior observer because the individual isounits may depend on time, $\hat{1}_k = \hat{1}_k(t, ...)$. To understand in full the implications, the reader should be aware that, when the isotopies of time of Class III are added, the interior observer can be arbitrarily in the future or in the past with respect to the exterior observer (see Sect. 3). Finally, if the internal units are altered *during* the observation, the exterior observer will see the isobox moving in space *without application of any force* (see later on in this section the geometric propulsion).

As a result of the above peculiar characteristics, far away stars which have a large distance from Earth when represented in Euclidean space, can have a distance as small or as large as desired when represented in isoeuclidean space. This notion is illustrated with the following self-evident property.

Definition 2.7: *The "geometric propulsion" is the mathematical displacement from a point* $P_1(x_1, y_1, z_1)$ *to a point* $P_2(x_2, y_2, z_2)$, *here assumed to be on the same straight line from the origin* 0 *in Euclidean space realized via such an isotopy of the underlying Euclidean geometry for which the isodistance* $D_{01}\times\hat{1}$ *between* 0 *and* P_1 *is such that* D_{01} *is equal to the distance* d_{02} *between* 0 *and* P_2 *(see Fig. 5.2.2 for details), i.e.*

$$\hat{D}_{01} = D_{01} \times \hat{1} \equiv d_{02} \times \hat{1}. \tag{2.2.13}$$

The equation of an *isostraight isoline* is given by one of the following forms

$$\hat{a} \hat{\times} \hat{x} + \hat{b} \hat{\times} y + \hat{c} \hat{\times} \hat{z} + \hat{d} = (a x + b y + c z + d) \times \hat{1} = 0,$$

$$\begin{cases} \hat{x} - \hat{x}_1 - \hat{p} \hat{\times} \hat{a}_1 = (x - x_1 - p a_1) \times \hat{1} = 0, \\ \hat{y} - \hat{y}_1 - \hat{p} \hat{\times} \hat{a}_2 = (y - y_1 - p a_2) \times \hat{1} = 0, \end{cases}$$

on a straight line from the origin 0 as in the figure. Let d_{01} and d_{02} be the conventional distances of the points P_1 and P_2 from the origin

$$d_{01} = (x_1^2 + y_2^2 + z_1^2)^{1/2}, \quad d_{02} = (x_2^2 + y_2^2 + z_2^2)^{1/2}, \quad d_{02} > D_{01}. \qquad (1)$$

The geometric propulsion of the point P_1 to P_2 occurs under the following steps: 1) the geometry underlying the point P_1 is lifted isotopically with resulting isodistance (5.2.9); 2) the isotopy is chosen according to law (.5.2.11), i.e.,

$$\hat{D}_{01} = (x_1^2 b_1^2 + y_1^2 b_2^2) \times \hat{1} = d_{02} \times \hat{1}, \qquad (2)$$

with simplest possible solution

$$b_1^2 = x_2^2 / x_1^2, \quad b_2^2 = y_2^2 / y_1^2, \quad b_3^2 = z_2^2 / z_1^2 \ \text{Det.}\hat{1} > 1; \qquad (3)$$

and 3) the geometry is then returned to the original Euclidean form. Under the above assumptions, the projection of the isopoint $\hat{P}_1(\hat{x}_1, \hat{y}_1, \hat{z}_1)$ in the Euclidean plane coincides with $P_2(x_2, y_2, z_2)$. The activation and subsequent de-activation of the isotopy then yield the motion from P_1 to P_2. It should be stressed that the above geometric propulsion is a *purely mathematical notion*, here defined for a *point*. The possibility of its future realizations is studied in Vol.s II and III and essentially deal with the identification of means which can alter the basic units of space (and time). The reader should keep in mind the most intriguing property of the geometric propulsion, that of being permitted by the *Euclidean* axioms themselves, only realized in a way more general than the usual one. Thus, an outside observer will simply see the motion from the point P_1 to P_2 *without any visible change of the geometry or visible means for the displacement.* As we shall see in the next section, even more intriguing properties emerge when introducing the geometric propulsion in space-time.

It is important to understand that while isoline (.5.2.13) is isostraight in isoeuclidean spaces, it is generally *curved* when projected in the conventional space. This property can be best inspected via an "old trick" of the isotopies, the reduction of the *isospace* $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ to a *conventional* space $\bar{E}(\bar{r},\delta,R)$ possessing the same invariant. This is readily possible for the values $\bar{r}_k = r_k b_k$ under which

$$r^i \times \hat{\delta}_{ij} \times r^j = \bar{r}^i \times \delta_{ij} \times \bar{r}^j. \qquad (2.2.16)$$

angular isotopic element and *angular isounits*, for the case of realization (2.60), are given respectively by

$$\hat{T}_{\hat{\alpha}} = b_1 b_2 , \qquad \hat{I}_{\hat{\alpha}} = b_1^{-1} b_2^{-1}. \tag{2.2.20}$$

The mechanism of isotopies of angles is therefore that a given angle α is lifted in the amount $\alpha \to \hat{\alpha} = \hat{T}_{\hat{\alpha}}\alpha$, but the unit is lifted by the inverse amount, $I \to \hat{I}_{\alpha} = \hat{T}^{-1}$, thus allowing the preservation of trigonometric axioms (App. A).

We shall say that two isovectors originating from $\hat{0}$ to the isopoints $\hat{P}_1$ and $\hat{P}_2$ in the isoplane $\hat{z} = \hat{0}$ are *isoperpendicular* when their intersecting isoangle is $\hat{\alpha}$ = 90°, which can hold iff

$$x_1 b_1^2 x_2 + y_1 b_2^2 y_2 \equiv 0 , \tag{2.2.21}$$

and they are *isoparallel* when their intersecting isoangle is null, $\hat{\alpha}$ = 0,which can hold iff

$$x_1 b_1^2 y_2 - y_1 b_2^2 x_2 \equiv 0 . \tag{2.2.22}$$

The above two conditions establish the existence of simple yet, unique and unambiguous isotopic images of the Euclidean axioms of perpendicularity and parallelism. It is then easy to prove the following properties.

Theorem 2.1. *The isotopies map perpendicular lines into isoperpendicular isolines and parallel lines into isoparallel isolines.*

By using these results, it is possible to prove that the isoeuclidean geometry with diagonal Class I isounits is expressible via the following main assumptions (see, e.g., ref. [43], Ch. 2, for a recent study of the conventional Euclidean axioms).

Isoaxiom I: *There exists one and only one isostraight isoline from one*

unknown at this writing.

$$\cos\alpha = \frac{x_1 x_2 + y_1 y_2}{(x_1 x_1 + y_1 y_2)^{\frac{1}{2}}(x_2 x_2 + y_2 y_2)^{\frac{1}{2}}} \quad . \tag{1}$$

In the transition to the Isoeuclidean plane, straight lines are generally mapped into curves when defined on the original plane over the conventional field R, as depicted in Diagram (B), thus implying the general loss of the notion of angle as typical in the transition from the Euclidean to the Riemannian geometry. The isotopies permit the reconstruction of the notion of angle, but only in the isoplane over isofields, because in the latter case the original straight lines are mapped into isostraight isolines as depicted in Diagram (C). In the latter case, the original angle α is lifted into the expression $\hat{\alpha} = b_1 b_2 \alpha$ called *isoangle* which is derived from the underlying Lie–Santilli isosymmetry of the isoplane $S\hat{O}(2)$ studied in detail in ref. [10]. Expression (1) is lifted into expression (5.2.16) which evidently does not characterize cosα any more and it is assumed as the definition of the isocos$\hat{\alpha}$, and studied in App. A.

The lifting of the additional axioms of the Euclidean geometry [loc. cit.] is left to the interested reader. Additional axiomatic properties are studied in App. A.

Isotopies characterized by *nondiagonal* isotopic elements are vastly unknown at this writing. We merely indicate that they imply a structural alteration of the original geometry more profound than that of diagonal isotopic elements. As an illustration, consider the isoeuclidean space as in Eq.s (5.2.5) but with isotopic element

$$\hat{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \tag{2.2.23}$$

for which $\det \hat{T} = 1$. Thus, $\hat{T}$ is nondiagonal but still of Class I.

It is easy to see that in this case the basic isoinvariant (5.2.5d) becomes identically null, i.e.,

$$\hat{r}^{\hat{2}} = (r^i \times \hat{\delta}_{ij} \times r^j) \times \hat{1} = (x\,x + y\,z - z\,y) \times \hat{1} \equiv (x\,x) \times \hat{1}, \tag{2.2.24}$$

namely, the isotopy is regular, thus invertible, yet the isospace is degenerate and

The isoeuclidean geometry of Class I is used in physics for the characterization of matter, while its isodual image of Class II is used for the characterization of *antimatter* [9,10]. The isogeometry ofg Class III appears to be the version particularly suitable for applications in theoretical biology.

Even though there are mathematical similarities, the reader should be aware of considerable differences between the isoeuclidean geometry of Class III suggested for theoretical biology, and the union of the isogeometries of Classes I and II used in physics. For these reasons, it is recommendable to study first the isodual isoeuclidean geometry of Class II per se, and then study the isogeometry of Class III.

As it is well known, the mere reversal of the sign of time leads to notorious and visible problematic aspects, such as the violation of causality, because it implies that effects can precede their causes. the *sole* alternative known to this author for avoiding these problematic aspects, is the map of the *entire* geometry into an *antiautomorphic* image.

The *sole* antiautomorphic map known to this author which is applicable beginning at the *classical* Euclidean and isoeuclidean levels is the so–called *isoduality,*

$$\hat{1} \rightarrow \hat{1}^d = -\hat{1} . \tag{2.2.25}$$

introduced in paper [27] of 1985.

Definition 2.8 [27]: *The "isodual isoeuclidean spaces" are given by the isoduals of the original isovectors* $\hat{r}^d = -\hat{r} = \{\hat{x}^d, \hat{y}^d, \hat{z}^d\} = \{-\hat{x}, -\hat{y}, -\hat{z}\}$, *called "isodual isovectors", defined on the isodual isospaces* $\hat{V}^d(\hat{r}^d,+,\hat{\odot}^d, \hat{R}^d(\hat{n}^d,+,\hat{x}^d))$ *over the isodual isofield* $\hat{R}^d(\hat{n}^d,+,x^d)$ *with isodual isounit* $\hat{1}^d = -\hat{1} < 0$ of Class II *(Sect. I.2.2) equipped with the original sum* + *and an isodual isoproduct* $\hat{\odot}^d = \odot^d\hat{T}^d\odot^d = \odot\hat{T}^d\odot = -\odot\hat{T}\odot$, $\hat{1}^d = (\hat{T}^d)^{-1} < \hat{0}$, *verifying the isodual images of properties 1)–14) of Definition 5.2.2 here omitted for brevity. The "isodual isoeuclidean geometry" is the geometry of the isodual isoeuclidean spaces of Class II. The "isoeuclidean geometries of Classes III, IV and V" are the geometries of isospaces with isounits of the*

Proposition 2.2: *The basic invariants of the Euclidean or isoeuclidean geometries are "isoselfdual", i.e.., invariant under isodualities, i.e.,*

$$r^2 \equiv r^{d2d}, \quad \text{and} \quad \hat{r}^{\hat{2}} \equiv \hat{r}^{d2d}. \tag{2.2.29}$$

As we shall see, the above mathematically elementary property has rather important physical implications for the representation of antimatter.

The *isodual isostraight line* can be expressed by

$$\hat{a}^d \hat{\times}^d \hat{x}^d + \hat{b}^d \hat{\times}^d \hat{y}^d + \hat{c}^d = (a \times x + b + y + c) \times \hat{1}^d = 0. \tag{2.2.30}$$

where $\hat{a}^d, \hat{b}^d, \hat{c}^d \in \hat{\mathbb{R}}^d$. A given straight line can therefore be also interpreted as belonging to an isospace *as well as* to its isodual. As we shall see, this additional elementary property is evidently extendable to curves and seems to have significant application in theoretical biology. In fact it indicates that, even though an object is *perceived* as belonging to our Euclidean geometry, and it *appear* to evolve with our time it may eventually belong to a structurally more general geometry with an inverted direction of time [30].

The *isodual angle* is the angle between two intersecting isodual straight lines in the isodual Euclidean plane, and it is simply given by $\alpha^d = -\alpha$. The *isodual isoangle* is the angle between two intersecting isodual isostraight isolines, and can be written

$$\hat{\alpha}^d = b_1^d b_2^d \alpha^d = -\hat{\alpha}. \tag{2.2.31}$$

We leave for brevity to the interested reader the definition of *isoperpendicular and isoparallel isodual isostraight isolines*, and the isodualities of the remaining properties of the isoeuclidean geometry.

The causal characterization of motion backward in time via the isodual geometry is made possible by the property of Proposition 2.2 and those of the following:

methods for its proper handling. To avoid prohibitive and perhaps premature technical difficulties, we shall essentially study hereon on the isogeometry of Class IV in which the zeros of the isounits have been removed, which can be treated separately as the solutions of the equations $\hat{1} = 0$.

We reach in this way a most important difference between the isogeometries used in physics and the broader geometries used in theoretical biology. The former are given by the isogeometries of Class I and, separately, those of Class II, while the latter are given by the isogeometry of Class IV consisting of the union *plus* their continuous interconnection.

The above lines are tacitly implied hereon whenever studies isodual theories.

2.2.D: Operations on isovectors and their isoduals.

We consider now the operations of isovectors in the isoeuclidean geometry with diagonal isounit, Eq.s (2.47). As proved in Ch. I.3, ref. [11], the basis of a vector space is not changed under isotopy (up to possible renormalization factors). Let e_k, k = 1, 2, 3, be the unit vectors of a three–dimensional Euclidean space E(r,δ,R) directed along the x, y, z axes, and let

$$\hat{e}_k = e_k \times \hat{1} , \tag{2.2.32}$$

be the corresponding *isobasis* in $\hat{E}(\hat{r},\hat{\delta},\hat{R})$. Then, a *isovector* $\hat{V}$ can be expressed in isospace

$$\hat{V} = x\,\hat{e}_1 + y\,\hat{e}_2 + z\,\hat{e}_3 . \tag{2.2.33}$$

This is another way of expressing the fact that the isovector $\hat{V}$ is isostraight in $\hat{E}(\hat{r},\hat{\delta},\hat{R})$, although its projection in E(r,δ,R) is curved. As expected, *the operations on vectors are preserved under isotopies.* In fact, the familiar scalar product of two vectors $V_1 = (x_1, y_1, z_1)$ and $V_2 = (x_2, y_2, z_2)$

$$\hat{V}_3 = \hat{V}_1 \hat{\wedge} \hat{V}_2 , \ \hat{V}_{3k} = \epsilon_{kij} (b_i x_{i1}) (b_j x_{2j}) , \ i, j, k = 1, 2, 3 . \qquad (2.2.41)$$

which satisfies the basic axioms of a vector product

$$\hat{V}_1 \hat{\wedge} \hat{V}_2 = \hat{V}_2 \hat{\wedge} \hat{V}_1 , \quad \hat{V}_1 \hat{\wedge} (\hat{V}_2 + \hat{V}_3) = \hat{V}_1 \hat{\wedge} \hat{V}_2 + \hat{V}_1 \hat{\wedge} \hat{V}_3 . \qquad (2.2.42)$$

Other operations on isovectors can be constructed accordingly.

The operations on isodual isovectors $\hat{V}^d = - \hat{V}$ on isodual spaces $\hat{E}^d(\hat{r}^d,\hat{\delta}^d,\hat{R}^d)$ with diagonal isodual isounits are easily derivable via the isodual map. As an example, the *isodual isoscalar product* is given by

$$\hat{V}_1^d \hat{\circ}^d \hat{V}_2 = (- x_1 b_1^2 x_2 - y_1 b_2^2 y_2 - z_1 b_3^2 z_2) \times \hat{1}^d \in \hat{R}^d , \qquad (2.2.43)$$

and it is manifestly isoselfdual.

Similarly, the *isodual isovector product* is given by

$$\hat{V}_3^d = \hat{V}_1^d \hat{\wedge}^d \hat{V}_2 , \ \hat{V}_{3k}^d = \epsilon_{kij} (b_i^d x_{i1}^d) \times^d (b_j^d x_{2j}^d) = - \hat{V}_{3k} . \qquad (2.2.44)$$

It is instructive for the interested reader to verify the preservation of *Lagrange's identity* under isotopies among four isovectors $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ in $\hat{E}(\hat{r},\hat{\delta},\hat{R})$

$$(\hat{A} \hat{\wedge} \hat{B}) \hat{\circ} (\hat{C} \hat{\wedge} \hat{D}) = (\hat{A} \hat{\circ} C) \hat{\times} (B \hat{\circ} \hat{D}) - (\hat{B} \hat{\circ} C) \hat{\times} (\hat{A} \hat{\circ} \hat{D}) . \qquad (2.2.45)$$

Other properties can be easily derived by the interested reader via similar procedures.

2.2.E: Representation of biological structures as isospheres.

We now pass to the study of another important geometric notion of this analysis, *the representation of biological structures as isospheres in isoeuclidean*

The ordinary sphere, hereon written in the form[5]

$$r^2 = (x^2 + y^2 + z^2) \times I = R^2 \times I \in R \tag{2.2.50}$$

is a trivial particular case of the isosphere of Class I. The "isodual sphere" is the image of the sphere under duality with equation

$$r^{d2d} = -x^2 - y^2 - z^2) \times I^d = R^2 \times I^d, \tag{2.2.51}$$

and it is also a particular case of the isodual isosphere of Class II.

The verification of the perfect spheridicity of the isosphere in isospace is important. Recall that, by central assumption, the Euclidean space and related Cartesian coordinates admit *the same unit for all three axes,* which is geometrically expressed by the unit I = diag. (1, 1, 1) of the basic SO(3) symmetry, and we shall write

$$I_x = I_y = I_z = +1 . \tag{2.2.52}$$

Recall that, also by central assumption, the isoeuclidean space and related isocartesian coordinates admit *different units for different axes* which can be expressed via the isounit $\hat{1}$ = diag. $(b_1^{-2}, b_2^{-2}, b_3^{-2})$ of the basic isosymmetry $S\hat{O}(3)$ (Ch. II.6), and we shall write

$$\hat{1}_x = b_1^{-2} \neq \hat{1}_y = b_2^{-2} \neq \hat{1}_z = b_3^{-2} \neq +1 . \tag{2.2.53}$$

Recall finally that the original geometric characteristics are preserved under the above an isotopy, e.g., a straight line is mapped into an isostraight isoline.

[5] Note that the formulation requires the redefinition of the ordinary field of real numbers with respect to the unit I = diag. (1, 1, 1). Such a reformulation is here necessary to have the ordinary sphere admitted as a particular case of the isosphere.

Proposition 2.4: *The maps from the sphere to the isodual sphere*

$$r^2 = (x^2 + y^2 + z^2) \times I \; = \; R^2 \times I \; \rightarrow \; r^{d2d} = (-x^2 - y^2 - z^2) \times I^d = R^2 \times I^d \, , \qquad (2.2.56)$$

and from the isosphere to the isodual isospheres

$$\hat{r}^{\hat{2}} \; = \; (r^i \times \hat{\delta}_{ij} \times r^j) \times \hat{1} = R^2 \times \hat{I} \; \rightarrow \; \hat{r}^{d2d} \; = \; (r^{di} \times \hat{\delta}^d{}_{ij} \times r^{dj}) \times \hat{1}^d = R^2 \times \hat{I}^d \, , \qquad (2.57)$$

are antiautomorphic, thus implying the reversal of the sign of all the original positive-definite characteristics.

The above property evidently permits the characterization of anti-hadrons as isodual isospheres. Note that in contemporary particle physics both hadrons and anti-hadrons are treated with the *same* geometry and are thus represented with the *same* sphere, resulting in the same classical characteristics which is basically insufficient for their distinction.

THE ISOSPHERE

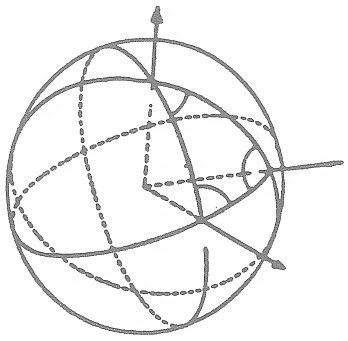


FIGURE 2.4: A schematic view of the perfect sphere in isospace over isofield introduced by this author [15] under the name of *isosphere*. Besides the

functions of App. A. We should finally mention that the symmetry of the isosphere is the *isorotational symmetry* outlined in App. C.

Proposition 2.5: *The sphere and the isosphere are isoselfdual*

$$r^2 = (x^2 + y^2 + z^2) \times I \equiv r^{d2d} = (-x^2 - y^2 - z^2) \times I^d,$$

$$\hat{r}^{\hat{2}} = (r^i \times \delta_{ij} \times r^j) \times \hat{1} = R^2 \hat{\times} \hat{1} \equiv \hat{r}^{d2d} = (r^{di} \times \delta^d{}_{ij} \times r^{dj}) \times \hat{1}^d = R^2 \hat{\times} \hat{1}^d, \qquad (2.2.58)$$

The above property is important for the observability of antihadrons in our space. In fact, it establishes that a given sphere cannot be claimed to belong to a particle or to an antiparticle without additional information, e.g., on charge, energy, etc. Additional properties of the isosphere will be studied in the next section and in Vol.s II, and III.

The following mathematical properties of the isosphere are self–evident.

Theorem 2.2 [15]: *The isosphere of Class III unifies all the following quadrics of the conventional Euclidean space*

1) All ordinary sphere

$SO(3)$: $x^1 x^1 + x^2 + x^2 + x^3 x^3 = R^2$,

2) All elliptic paraboloids *(paraboloids with one sheet)*

$SO(2.1)$: $x^1 x^1 - x^2 x^2 + x^3 x^3 = R^2$,

3) All prolate or oblate ellipsoids

$\hat{SO}(3)$: $x^1 b_1^2 x^1 + x^2 b_2^2 x^2 + x^3 b_3^2 x^3 = R^2$,

4) All isotopic deformations of the elliptic paraboloids

The *isosphere of Class V* has not been investigated to date, and it is expected to permit the formulation of new notions of "spheres", such as spheres whose radius is a step function or a lattice.

ISOTOPIC UNIFICATION OF QUADRICS

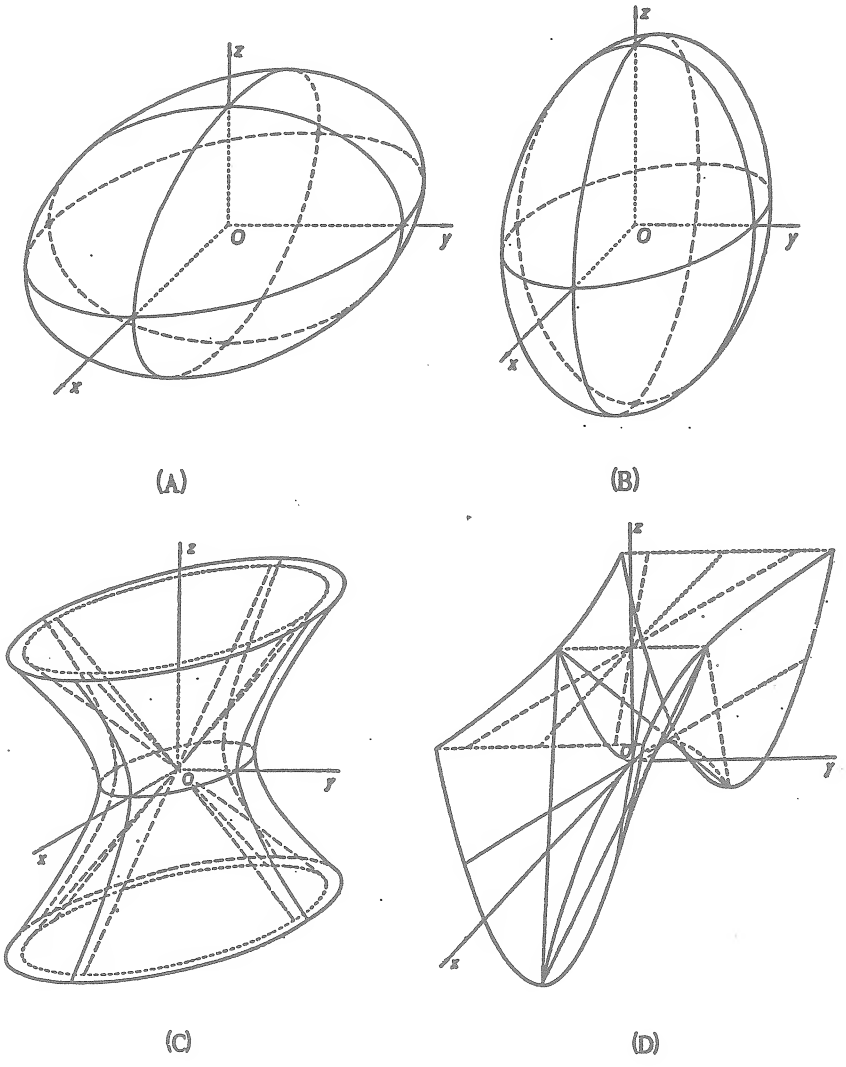


FIGURE 2.5: A schematic view of the unification into the isosphere of Class III of prolate ellipsoid (A), oblate ellipsoid (B), one sheet hyperboloid (C) and two sheets

Lemma 2.2: *Isoeuclidean spaces are curved unless the isometric is independent on the local coordinates, but dependent on the remaining variables,* $\hat{\delta} = \hat{\delta}(t, \dot{r}, \ddot{r}, ...)$.

Proof. A given n–dimensional isoeuclidean space $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ admits the *non–null* Christoffel symbols (connection)

$$\hat{\Gamma}_{h\ k}^{\ i} = \frac{1}{2}\hat{\delta}^{ij}\left(\frac{\partial\,\hat{\delta}_{kj}}{\partial\, r^h} + \frac{\partial\,\hat{\delta}_{jh}}{\partial\, r^k} - \frac{\partial\,\hat{\delta}_{hk}}{\partial\, r^j}\right), \tag{2.2.61}$$

which characterize the quantities

$$\hat{R}_{l\ h}^{\ j} = \frac{\partial\hat{\Gamma}_{l\ h}^{\ j}}{\partial r^k} - \frac{\partial\hat{\Gamma}_{l\ k}^{\ j}}{\partial r^h} + \hat{\Gamma}_{q\ k}^{\ j}\,\hat{\Gamma}_{l\ h}^{\ q} - \hat{\Gamma}_{q\ h}^{\ j}\,\hat{\Gamma}_{l\ k}^{\ q}, \tag{2.2.62}$$

representing non–null curvature, which is identically null when the isometric is independent from the local coordinates. **q.e.d.**

In short, *the isoeuclidean geometry provides a symbiotic unification of the Euclidean and Riemannian geometries.* The curved character of the isoeuclidean geometry has been computed in the projection of the isospace in the original space. Nevertheless, as we shall see later on in this chapter, the above curvature persists even in isospace.

This author must admit that the emergence of curvature on an isospace which is isoflat was basically unexpected and, for this reason, it was identified for the first time only after some twelve years following the identification of the isoeuclidean geometry [28]. As we shall see, the property is nontrivial, inasmuch as, when extended to the isotopies of our space–time, it permits a geometric unification of general and special relativities with ensuing operator form of gravity with curvature embedded in the unit of relativistic quantum theories.

The curved character of the isoeuclidean geometry when projected on conventional spaces over conventional fields can be studied via the same methods used for the Riemannian geometry, only referred to a different unit and it will not be studied here for brevity.

A few comments are in order on the comparison of the isoeuclidean

Euclidean geometry preserving straight lines and circles under the conventional value of the unit; while

B) *the isoeuclidean geometry identifies "an infinite class" of liftings of the Euclidean geometry which preserve straight lines and circles under a joint lifting of the unit.*

Note finally that the Lobacevski geometry itself can be subjected to an isotopic lifting which has not been studied here for brevity.[6]

Numerous other noneuclidean geometries exist in the literature (besides the Minkowskian [38], symplectic [39], and Riemannian [40] geometries. One particularly intriguing geometry is the so–called *nondesarguesian geometry* studied by Shoeber [41], which has a significant connection with these because it is also capable of representing variationally nonselfadjoint (that is, nonhamiltonian) systems. As such, the nondesarguesian geometry is capable of representing some (although not all of) the conditions presented in Sect. 1.B for the characterization of biological systems, such as nonconservative character and irreversible structure.

This latter geometry too is different from the isoeuclidean one, again, because it is based on the conventional unit. However, the underlying mapping between the Euclidean and nondesarguesian geometry is also contained as a particular case of the infinite transformations (5.2.72) of the isoeuclidean geometry.

A similar situation also occurs for the *Finslarian geometry* [40] which is particularly suited to represent *anysotropic* structures due to a preferred direction, which is usually reached via the factorization of such direction in the metric itself. As such, the Finslerian geometry is also significant for the representation of anisotropic features in biological structures. However, it is easy to see that the Finslerian geometry are a particular case of the isoeuclidean geometry, the latter being much broader in structure and content.

These comments are significant to focus the attention on an additional reason for our selection of the isoeuclidean geometry over other possible choices, its "direct universality" that is, its capability of incorporating "all" infinitely

[6] Note that the *isolobacevki geometry* is no longer contained as a particular case of the isoeuclidean geometry because the original axioms of the two geometries are different.

which are defined in the space

$$\hat{S}(t, r, v) = E(t)\times E(r,\delta,R)\times E(v,\delta,R), \tag{2.3.2}$$

where $E(v,\delta,R)$ is the *tangent space* of $E(r,\delta,R)$, or in their equivalent first-order (vector-field) form

$$(db^{\mu} / dt) = \begin{pmatrix} dr^{ka} / dt \\ dp_{ka} / dt \end{pmatrix} = (\Xi^{\mu}) = \begin{pmatrix} v_{ka} = p_{ka} / m_a \\ F^{SA}{}_{ka}(t, r, p) + F^{NSA}{}_{ka}(t, r, p) \end{pmatrix} \tag{2.3.3}$$

which are defined over the space

$$S(t,r,p) = E(t)\times E(r,\delta,R)\times E(p,\delta,R), \tag{2.3.4}$$

where $E(p,\delta,R)$ is the cotangent space to $E(r,\delta,R)$.

As it is well known, conventional analytic methods are derivable from the *canonical first-order action principle*

$$\delta\int_{t_1}^{t_2} L(t, x, v)\, dt = \delta\int_{t_1}^{t_2} [\, p_k dx^k - H(t, x, p)dt] = \delta\int_{t_1}^{t_2} [\, R^{\circ}_{\mu}(b)db^{\mu} - H(t, b)dt] = 0,$$

$$R^{\circ} = \{ R^{\circ}_{\mu} \} = \{ p_k, 0 \}, \qquad k = 1, 2, ..., N, \qquad \mu = 1, 2, ..., 2N . \tag{2.3.5}$$

The variation of the above action then yields the contemporary formulation of Lagrange equations along an actual path P°

$$\left\{ \frac{d}{dt}\frac{\partial L(t, r, v)}{\partial v^k} - \frac{\partial L(t, r, v)}{\partial r^k} \right\} (P^{\circ}) = 0 , \tag{2.3.6}$$

with corresponding contemporary form of Hamilton's equations in the unified notation $b = \{b^{\mu}\} = \{r^k, p_k\}$

$$\left\{ \omega_{\mu\nu}\frac{db^{\nu}}{dt} - \frac{\partial H(t, b)}{\partial b^{\mu}} \right\} (P^{\circ}) = 0 , \tag{2.3.7}$$

Under certain continuity and regularity conditions here ignored for simplicity [4,5], the latter approach does indeed permit the achievement of the so–called *indirect universality*, that is, the representation of all possible Newtonian systems (3.1) (universality) in an equivalent coordinate systems (indirect universality). This is the *Lie–Koening theorem* [5] as the analytic counterpart of the *Darboux's theorem* of the symplectic geometry.

The first basic insufficiency is that *the above indirect universality is solely applicable to local–differential systems, while the Lie–Koening and Darboux theorems cannot be even formulated for nonlocal–integral systems.*

But even when applicable to local–differential systems, the above indirect universality has rather serious drawbacks for applications. First, the transformations needed for the reduction of a nonhamiltonian system in a given frame to a Hamiltonian form in another frame are non highly noncanonical and linear and, as such, *the coordinates of the equivalent Hamiltonian form are not realizable in laboratory* (as an example, no biological observation can be done in a frame in which $r' = \exp(Nrv)$).

Also, their nonlinearity implies the *loss of the original inertial character of the reference frame, with consequential loss of conventional relativities* (in fact, the Galilei and Einstein relativities are solely applicable to inertial systems, as well known, thus preventing the use of the Lie–Koening and Darboux Theorems).

The above drawbacks imply the *fundamental condition* of our studies according to which *analytic methods must be selected in such a way to provide a direct representation, that is, a representation in the fixed coordinates of the observer.* Only after the achievement of this representation the use of the transformation theory *may* have a physical relevance.

The fundamental analytic problem studied by this author in monographs [4–5,7–8,9–10] can therefore be formulated as follows:

Fundamental analytic problem: *Identify generalized analytic methods which permit direct universality for all possible well behaved nonlinear, nonlocal and nonhamiltonian systems.*

where $\Omega_{\mu\nu}$ is an exact, nowhere degenerate (and therefore symplectic) tensor although in its most general possible realization

$$\Omega_{\mu\nu} = \partial_\mu R_\nu - \partial_\nu R_\mu . \qquad (2.3.11)$$

The conventional Hamilton's equations are an evident particular case for $R = R^\circ = (p, 0)$, as the reader can verify.

A "Theorem of Direct Universality" was proved in ref. [5], Sect. 4.5, according to which (again under certain topological conditions here ignored) Pfaffian actions (3.9) can represent all possible analytic, regular and variationally non–self–adjoint Newtonian systems (universality) directly in the frame of the experimenter (direct universality).

The above approach permitted the resolution of the major drawbacks in the use of the historical analytic equations with external terms, the loss of Lie's theory, the symplectic geometry and of the action principle. However, subsequent studied indicate that, despite its generalized character, Birkhoffian mechanics resulted to be insufficient for the desired objectives.

In fact, *Birkhoffian mechanics can only represent local–differential systems,* due to the strictly local–differential character of the underlying symplectic geometry and, as such, it is not suitable for the desired representation of nonlocal systems.

Moreover, the operator image of Birkhoffian mechanics resulted to be structurally beyond the guidelines of quantum mechanics (e.g., it would require a "wave function" $\psi(t, r, p)$ with a *necessary* joint dependence on r and p which is outside all tenets of quantum mechanics, see App. 2.B of ref. [10]).

In view of the above insufficiencies, after completing the rather laborious task of construction the step–by–step Birkhoffian generalization of Hamiltonian mechanics, this author was forced to seek more adequate analytic methods for the direct representation of well behaved, but otherwise arbitrary, linear and nonlinear, local and nonlocal and Hamiltonian or nonhamiltonian systems.

Such a *classical* solution was *necessary* for this author (a particle physicist) to initiate quantitative studies on the historical open legacy due to

2.3.B. Isotopic lifting of Newtonian methods

In this section we review the nonlinear, nonlocal and nonhamiltonian isotopies of Newton's equations as characterized by the isodifferential calculus and first presented in ref.s [32–35]. The isotopies have been selected over a variety of other possibilities because of their axiom–preserving character as well as of the consequential broadening of classical and quantum mechanics outlined in subsequent sections.

The contemporary formulation of Newton's equations, which as remained essentially that originally formulated by Newton in the 1600's, requires the tensorial space $S(t,x,v) = E(t)\times E(r,\delta,R)\times E(v,\hat{\delta},R)$ where $E(t)$ is the one–dimensional space representing time t, $E(r,\delta,R)$ is the conventional three–dimensional Euclidean space with local trajectories $r(t) = \{r^k\} = \{x, y, z\}$ and $E(v,\delta,R)$ is the tangent space TE which, at this Newtonian level, can be considered as an independent space representing the contravariant velocities $v = \{v^k\} = dr^k/dt$.

Newton's equations for one particle of mass m = const. ($\neq 0$) moving within a resistive medium (e.g., our atmosphere) can then be written

$$m\, dv_k / dt - F^{SA}{}_k(t, r, v) - F^{NSA}{}_k(t, r, v) = 0\,,\ k = 1, 2, 3\,(= x, y, z)\,, \quad (2.3.12)$$

where SA (NSA) stands for *variational self–adjointness* (*variational non–self–adjointness*), i.e.. the verification (violation) of the necessary and sufficient conditions for the existence of a potential U(t, r, v) originally due to Helmholtz (see monograph [7] for historical notes and systematic studies). It should be recalled that in Newtonian mechanics the potential U(t, r, v) must be linear in the velocities to avoid a redefinition of the mass,

$$U(t, x, v) = U_k(t, x)\, v^k + U_0(t, x)\,. \quad (2.3.13)$$

Eq.s (3.11) can then be written

$$\hat{1}_{tot} = \hat{1}_t \times \hat{1} \times \hat{1}, \tag{2.3.17}$$

For clarity, we continue to differentiate the *isotime* $\hat{t}$, *isocoordinates* $\hat{r}^k(\hat{t})$ and *isovelocities* $\hat{v}^k(\hat{t})$ from the original respective quantities t, r^k and v^k, with the following relationships

$$\hat{t} = t, \quad \hat{r}^k \equiv r^k, \quad \hat{r}_k = \hat{\delta}_{ki}\, \hat{r}^i \equiv r_k, \quad \hat{v}^k \equiv v^k,$$

$$\hat{v}_k = \hat{\delta}_{kj}\, \hat{v}^j = \hat{T}_k{}^i\, \delta_{ij}\, \hat{v}^j = \hat{T}_k{}^i\, v_i \neq v_k = \delta_{ki}\, v^i. \tag{2.3.18}$$

Also, we shall continue to ignore throughout this section the isoscalar character of the coordinates because expressible by the simple factorization of their respective units which then cancel out in the product, e.g., $\hat{r}\hat{\times}f = (r\times\hat{1})\times\hat{T}\times f \equiv r\times f$.

Definition 2.10 [32–35]: *The isotopic lifting of Newton's equations in isospace* $\hat{S}(\hat{t},\hat{r},\hat{v})$ *of Class III, here called "isotopic Newton equations", are given by*

$$\hat{\Gamma}_k(\hat{t}, \hat{r}, \hat{v}) = \hat{m}\frac{\hat{d}\hat{v}_k}{\hat{d}\hat{t}} - \frac{\hat{d}}{\hat{d}\hat{t}}\frac{\hat{\partial}\hat{U}(\hat{t}, \hat{r}, \hat{v})}{\hat{\partial}\,\hat{v}^k} + \frac{\hat{\partial}\hat{U}(\hat{t}, \hat{r}, \hat{v})}{\hat{\partial}\,\hat{r}^k} =$$

$$= \hat{m}\frac{\hat{d}\hat{v}_k}{\hat{d}\hat{t}} - \frac{\hat{\partial}\hat{U}_k(\hat{t}, \hat{r})}{\hat{\partial}\,\hat{r}^i}\frac{\hat{d}\hat{x}^i}{\hat{d}\hat{t}} + \frac{\hat{\partial}\hat{U}_o(\hat{t}, \hat{r})}{\hat{\partial}\,\hat{r}^k} = 0,$$

$$\hat{U}(\hat{t}, \hat{r}, \hat{v}) = \hat{U}_k(\hat{t}, \hat{r})\,\hat{v}^k + \hat{U}_o(\hat{t}, x\hat{r}), \tag{2.3.19}$$

where $\hat{m} = const\ (\neq 0)$ *is the "isotopic mass", that is, the image of the Newtonian mass in isospace and one should note the preservation of the linearity of isopotential* $\hat{U}$ *in* $\hat{v}^k$.

We are now equipped to prove the following:

Theorem 2.3 [loc. cit.]: *All possible sufficiently smooth, regular, but*

always admits a solution in the unknown quantities $\hat{m}$, $\hat{T}_t$, $\hat{T}_k^i$, $\hat{U}_k$ and $\hat{U}_o$ for given equations (3.14). In fact, system (3.21) is *overdetermined* and the following solution exists for *diagonal* space isounit and *constant* time isounit,

$$\hat{1}^i_k = \delta^i_k \, e^{f_k(t, x, v)}, \qquad \hat{1}^o_o = \text{constant} > 0 , \tag{2.3.23}$$

for which

$$\hat{m}\, T_t \equiv m , \quad \hat{U}_k(t, r) = \hat{1}_t \, U_k(t, r) , \quad \hat{U}_o(t, r) = U_o(t, r) ,$$

$$f_k(t, r, v) = - \, m^{-1} \int_o^t dt \, F^{NSA}{}_k(t, r, v) \, / \, v_k, \tag{3.2.24}$$

where there are no repeated indices, $\hat{m}$ is constant and the functions f_k are computed from Eq.s (3.22b). **q.e.d**

The primary motivations for the construction of the isotopic Newton's equations are expressed by the following properties with self-evident proofs.

Corollary 2.3.A [32-35]: *The isotopic Newton equations permit a representation of the actual nonspherical shape of the body considered and of all its possible deformations via the generalized unit (or isotopic element) of the theory.*

Recall that Newton's equations were based on the Galilean approximate the body considered as a massive point. The point-like representation of particles then implied only action-at-a-distance, potential interactions with consequential analytic representations via Hamilton's equations as well as under symplectic map into quantum mechanical formulations.

A representation of the extended character of particles in conventional methods is reached in *second* quantization via the *form factors*. However, this representation is restricted to spherical shapes from the fundamental symmetry of quantum mechanics, the rotational symmetry O(3). Moreover, the latter

Moreover, the *nonspherical character of the shape emerges only in the projection in ordinary spaces, because at the isotopic level all particles are represented via the isosphere, i.e., the perfect sphere in isospace* (Sect. II.2),

$$\hat{r}^2 = (x^1 n_1^{-2} x^1 + x^2 n_2^{-2} x^2 + x^3 n_3^{-2} x^3) \times \hat{1} \in \hat{R}(\hat{n},+,\hat{x}) . \qquad (2.3.27)$$

The representation of shapes more complex than the spheroidal ellipsoids is possible with non-diagonal isounits. The representation of the *deformations* of the original shape due to motion within resistive media or other reasons, can be achieved via a suitable functional dependence of the $\hat{T}_k^{\ i}$ terms in velocities, pressure, etc. (see [7-10] for various applications in classical and quantum mechanics).

Corollary 2.3.B: *The isotopic Newton equations permit a novel representation of nonpotential, variationally nonselfadjoint forces via the isometric of the underlying geometry, according to the rules*

$$m\, dv_k / dt - F^{NSA}{}_k(t, r, v) \equiv \hat{1}^i{}_k\, m\, d\, \hat{T}_i^{\ j} v_j / dt , \qquad (2.3.28)$$

while leaving unchanged the representation of conventional self-adjoint forces (except for the constant factor $\hat{T}_t$ *of* U_k*).*

In fact, the non-self-adjoint forces are embedded in the *covariant* coordinates in isospace $\hat{v}_i = T_i^{\ j} v_j$, where the v_j are the covariant coordinates in conventional space. The novelty therefore lies in the fact that *nonselfadjoint forces are represented by the isogeometry itself*, thus providing another motivation for the isotopies.

The simplicity of representation (3.21) should be kept in mind and compared to the complexity of the conventional solution of the *inverse problem of Newtonian mechanics* [4,5], i.e., the representation of nonselfadjoint systems via a Lagrangian or a Hamiltonian. Moreover, under the assumed conditions, the latter exists in the fixed coordinates (t, r, v) of the observer only for a restricted class called *nonessentially nonselfadjoint* [loc. cit.], while isorepresentation (3.21)

the isounit of the theory.

The above occurrence is permitted by Tsagas–Sourlas integro–differential topology of isomanifolds $\hat{M}(\hat{E})$ reviewed in Sect. II.1. Consider as an example the integro–differential equation

$$m\, dv/dt + \gamma\, v^2 \int_\sigma d\sigma\, \mathcal{F}(\sigma, ...) = 0, \qquad \gamma > 0, \tag{2.3.33}$$

representing an extended object (such as a space–ship during re–entry in our atmosphere) with local–differential center–of–mass trajectory r(t) and corrective terms of integral type due to the shape (surface) σ of the body moving within a resistive medium, where $\mathcal{F}$ is a suitable kernel depending on σ as well as on other variables such as pressure, temperature, density, etc. The above equation admits isorepresentation (3.21) with the values

$$\hat{T} = \hat{S}_\sigma\, e^{\gamma\, m^{-1}\, r \int_\sigma d\sigma\, \mathcal{F}(\sigma, ...)}, \quad \hat{T}_t = 1, \quad U_k = U_o = 0, \tag{2.3.34}$$

where $\hat{S}_\sigma$ is the shape factor, which is admitted by the integro–differential topology of the isomanifold $M(\hat{E})$ because all integral terms are embedded in the isounit. Similar isorepresentations can be easily constructed by the interested reader.

It should be recalled that the representation of nonlocal–integral terms is prohibited in Hamiltonian mechanics because the underlying geometry and topology are local–differential. Moreover, whether in physics or biology, nonlocal–integral effects are of contact zero–range type. Their representation (which can be sometime seen in the literature) via "integral potentials" has no mathematical or physical sense of any nature.

As a final remark, the reader should be aware that the isonewton's equations have been formulated here for Class III, thus including those for Class I, $\hat{1} > 0$, and those of Class II, $\hat{1} < 0$. The former are based on a positive–definite module and, therefore, all characteristics are positive–definite as in conventional

smooth and regular in a star-shaped region D *of their variables can always be identically rewritten as first-order action isofunctionals in isospace of Class III* $\hat{E}(\hat{t})\times\hat{E}(\hat{r},\hat{\delta},\hat{R})\times\hat{E}(\hat{v},\hat{\delta}\hat{R})$ *which are bilinear in the velocities,*

$$\hat{A} = \int_{t_1}^{t_2} dt\, \mathcal{L}(t, x, v, a, ...) = \hat{\int}_{\hat{t}_1}^{\hat{t}_2} \hat{dt}\, \hat{L}(\hat{t}, \hat{x}, \hat{v}),$$

$$\hat{L} = \tfrac{1}{2}\, \hat{m}\, \hat{v}^i\, \hat{\delta}_{ij}\, \hat{v}^j - \hat{U}^i(\hat{t}, \hat{x})\, \hat{\delta}_{ij}\, v^j - U_o(\hat{t}, \hat{x}) =$$

$$= \tfrac{1}{2}\, \hat{m}\, \hat{v}_k\, \hat{v}^k - \hat{U}_k(\hat{t}, \hat{x})\, v^k - \hat{U}_o(\hat{t}, \hat{x}), \qquad (2.3.37)$$

In fact, identities (3.35a) are overdetermined because, for each given $\mathcal{L}$, there exist infinitely many choices of $\hat{m}$, $\hat{T}_t$, $\hat{T}_i^j$, $\hat{U}_k$ and $\hat{U}_o$. We shall assume that integral terms are admitted in the integrand provided that they are *all* embedded in the isometric.

The *isovariational calculus* is a simple extension of the isodifferential calculus. In fact, we can write the following isovariation along an admissible isodifferentiable path $\hat{P}$

$$\hat{\delta}\, \hat{A}(\hat{P}) = \hat{\int}_{\hat{t}_1}^{\hat{t}_2} \hat{dt}\, \{ \hat{\delta r}^k \frac{\hat{\partial}}{\hat{\partial r}^k} + \hat{\delta v}^k \frac{\hat{\partial}}{\hat{\partial v}^k} \}\, \hat{L}(\hat{P}) =$$

$$= \hat{\int}_{\hat{t}_1}^{\hat{t}_2} \hat{dt}\, \{ \frac{\hat{\partial L}}{\hat{\partial r}^k} - \frac{\hat{d}}{\hat{dt}} \frac{\hat{\partial L}}{\hat{\partial v}^k} \}\, (\hat{P})\, \hat{\delta r}^k, \qquad (2.3.38)$$

where we have used isointegration by parts.

The above variational principle then yields the *isolagrange equations* first introduced in Ref.s [32-35]

$$\hat{L}_k(\hat{P}_o) = \left\{ \frac{\hat{d}}{\hat{dt}} \frac{\hat{\partial}\, \hat{L}(\hat{t},\hat{r},\hat{v})}{\hat{\partial}\, \hat{v}^k} - \frac{\hat{\partial}\, \hat{L}(\hat{t}, \hat{r}, \hat{v})}{\hat{\partial}\, \hat{r}^k} \right\} (\hat{P}_o) = 0, \qquad (2.3.39)$$

non-Newtonian forces, e.g., forces of integral type or acceleration-dependent. Discontinuous Newtonian forces, such as those of impulsive type, have been removed from the theorem because of lack of current knowledge on the topology of isospaces of Class V with discontinuous isounits, although such an extension is expected to exist, and its study is left to interested readers.

Note the simplicity of the construction of an isolagrangian representation as compared to the complexity of the construction of a conventional Lagrangian representation [4,5], when it exists.

Again, we have expressed Eq.s (3.39) in their Class III form which is the union of Class I and II. The equations of the latter class can be written in isodual isospaces $\hat{S}^d = \hat{E}^d(\hat{t}^d)\times\hat{E}^d(\hat{r}^d,\hat{\delta}^d,\hat{R}^d)\times\hat{E}^d(\hat{v}^d,\hat{\delta}^d,\hat{R}^d)$

$$\left\{ \frac{\hat{d}^d}{\hat{d}t^d} \frac{\hat{\partial}^d \hat{L}^d(\hat{t}^d,\hat{r}^d,\hat{v}^d)}{\hat{\partial}\,\hat{v}^{kd}} - \frac{\hat{\partial}^d \hat{L}^d(\hat{t}^d,\hat{r}^d,\hat{v}^d)}{\hat{\partial}\,\hat{r}^{kd}} \right\} (\hat{P}_o{}^d) = ,$$

$$= - \left\{ \frac{\hat{d}}{\hat{d}t} \frac{\hat{\partial}\,\hat{L}(\hat{t},\hat{r},\hat{v})}{\hat{\partial}\,\hat{v}^k} - \frac{\hat{\partial}\,\hat{L}(\hat{t},\hat{r},\hat{v})}{\hat{\partial}\,\hat{r}^k} \right\} (\hat{P}_o) = 0 , \qquad (2.3.41)$$

and they provide a direct analytic representation of the isodual Newton's equations.

2.3.D. Isotopies of Hamiltonian methods

We now study the *isotopies of the Legendre transform* based on the isodifferential calculus. For this purpose, we introduce the following isodifferentials in isospace $\hat{S}(\hat{t},\hat{r},\hat{p}) = \hat{E}(\hat{t})\times\hat{E}(\hat{r},\hat{\delta},\hat{R})\times\hat{E}(\hat{p},\hat{\delta},\hat{R})$

$$\hat{d}\,\hat{t} = \hat{1}_t\, dt , \quad \hat{d}\,\hat{r}^k = \hat{1}^k{}_i\, dr^i , \quad \hat{\partial}\,\hat{r}^i / \hat{\partial}\,\hat{r}^j = \hat{\delta}^i{}_j , \text{ etc.} ,$$

$$\hat{d}\hat{p}_k = \hat{T}_k{}^i\, d\hat{p}_i , \qquad \hat{d}\hat{p}^k = \hat{1}^k{}_i\, d\hat{p}^i , \qquad \hat{\partial}\hat{p}_i / \hat{\partial}\hat{p}_j = \hat{\delta}_i{}^j , \text{ etc.} \qquad (2.3.42)$$

with isounits

introducing the notation

$$\hat{R}^\circ = \{ \hat{R}^\circ_\mu \} = \{ \hat{p}_k, \hat{0} \}, \qquad \mu = 1, 2, ..., 2N, \qquad k = 1, 2, ..., N, \qquad (2.3.47)$$

the *isocanonical principle* assumes the form along an actual path $\hat{P}_o$

$$\hat{\delta}\, \hat{A}^\circ = \hat{\delta} \int_{\hat{t}_1}^{\hat{t}_2} \hat{dt}\, (\hat{p}_k\, \hat{dr}^k/\hat{dt} - \hat{H})\, (\hat{P}_o) = \hat{\delta} \int_{\hat{t}_1}^{\hat{t}_2} \hat{dt}\, (\hat{R}^\circ_\mu\, \hat{\partial}^\mu - \hat{H})\, (\hat{P}_o) =$$

$$= \int_{\hat{t}_1}^{\hat{t}_2} \hat{dt}\, \{ \hat{\delta p}_i \frac{\hat{\partial}}{\hat{\partial p}_i} + \hat{\delta v}^i \frac{\hat{\partial}}{\hat{\partial v}^i} + \hat{\delta r}^i \frac{\hat{\partial}}{\hat{\partial r}^i} \} (p_k\, v^k - H)\, (\hat{P}_o) =$$

$$= \int_{\hat{t}_1}^{\hat{t}_2} \hat{dt}\, [(\frac{\hat{dr}^k}{\hat{dt}} \frac{\partial \hat{p}_k}{\hat{\partial p}_i} - \frac{\partial \hat{H}}{\partial p_i}) \hat{\delta p}_i - [\frac{\hat{d}}{\hat{dt}} (\hat{p}_k \frac{\partial \hat{v}^k}{\partial \hat{v}^i}) + \frac{\partial \hat{H}}{\hat{\partial r}^i}) \hat{\delta x}^i]\, (\hat{P}_o) =$$

$$= \int_{\hat{t}_1}^{\hat{t}_2} \hat{dt}\, \{ \hat{\delta b}^\mu \frac{\hat{\partial}}{\hat{\partial b}^\mu} + \hat{\delta c}^\mu \frac{\hat{\partial}}{\hat{\partial c}^\mu} \} (\hat{R}^\circ_\mu\, \hat{db}^\mu - \hat{H}\, \hat{dt})\, (\hat{P}_o) =$$

$$= \int_{\hat{t}_1}^{\hat{t}_2} \{ \frac{\partial \hat{R}^\circ_\nu}{\partial \hat{b}^\mu} - \frac{\partial \hat{R}^\circ_\mu}{\partial \hat{b}^\nu}) \frac{\hat{db}^\nu}{\hat{\partial t}} - \frac{\partial \hat{H}}{\partial \hat{b}^\mu} \} (\hat{P}_o) \hat{\times} \hat{\delta b}^\mu = 0, \qquad (2.3.48)$$

The above principle yields the *isohamiltonian equations* of Class III introduced in ref.s [32–35], which can be written in conventional notation

$$\frac{\hat{dr}^k}{\hat{dt}} = \frac{\partial \hat{H}(\hat{t}, \hat{r}, \hat{p})}{\partial \hat{p}_k}, \qquad \frac{\hat{dp}_k}{\hat{dt}} = - \frac{\partial \hat{H}(\hat{t}, \hat{r}, \hat{p})}{\partial \hat{r}^k}, \qquad (2.3.49)$$

or in unified notation

$$\left(\frac{\partial \hat{R}^\circ_\nu}{\partial \hat{b}^\mu} - \frac{\partial \hat{R}^\circ_\mu}{\partial \hat{b}^\nu} \right) \frac{\hat{db}^\nu}{\hat{dt}} - \frac{\partial \hat{H}(\hat{t}, \hat{b})}{\partial \hat{b}^\mu} = 0, \qquad (2.3.50)$$

We now study the following additional property of isohamiltonian mechanics which is important for operator maps. The *isotopic Hamilton–Jacobi problem* (see, e.g., [5], p. 201 and ff. for the conventional case) is the identification of an isocanonical transform under which the Hamiltonian becomes null. The generating function of such a transform is the isocanonical action itself, resulting in the end–point contributions

$$\hat{d}\hat{A} = \hat{d}\int_{t_o}^{t} (\hat{p}_k\, \hat{d}\hat{x}^k - \hat{H}\, \hat{d}\hat{t}\,) = \Big| \hat{p}_k\, d\hat{x}^k - \hat{H}\, \hat{d}\hat{t} \Big|_{t_o}^{t}, \qquad (2.3.54)$$

with *isotopic Hamilton–Jacobi equations*

$$\frac{\hat{\partial}\hat{A}}{\hat{\partial}\,\hat{t}} + \hat{H}(\hat{t}, \hat{r}, \hat{p}) = 0\,, \qquad \frac{\hat{\partial}\hat{A}}{\hat{\partial}\hat{r}^k} - \hat{p}_k = 0\,. \qquad (2.3.55)$$

plus initial conditions $\hat{\partial}\hat{A}/\hat{\partial}\hat{r}^{\circ k} = \hat{p}_k{}^{\circ}$, where $\hat{r}^{\circ}$ and $\hat{p}^{\circ}$ are constants. The reader can easily work out the remaining properties of the isohamiltonian mechanics (see also [10]).

Note the *abstract identity between the conventional and isotopic mechanics* for all positive–definite isounits. In fact, in this case at the abstract level there is no distinction between dt and $\hat{d}t$, dr and $\hat{d}\hat{r}$, etc. The isolagrange and isohamilton equations therefore coincide at the abstract level with the conventional equations. This illustrates the axiom–preserving character of the isotopies, this time, at the analytic level.

The connection between the Birkhoffian and the isohamiltonian mechanics is intriguing. In fact, the Pfaffian action can always be identically rewritten as the isotopic action

$$\int_{t_i}^{t_2} [R_\mu(b)\, db^\mu - H(t, b)\, dt\,] \equiv \hat{\int}_{\hat{t}_i}^{\hat{t}_2} [\hat{R}^{\circ}_\mu(\hat{b})\, \hat{d}\hat{b}^\mu - \hat{H}(\hat{t}, \hat{b})\, dt\,],$$

$$\hat{b}^\mu \equiv b^\mu \quad \hat{H} \equiv H\,, \quad \hat{d}\hat{t} = dt, \qquad (2.3.56)$$

and the genèral, totally antisymmetric Lie tensor $\Omega^{\mu\nu}$ always admits the factorization into the canonical Lie tensor $\omega^{\mu\nu}$ and a regular symmetric matrix

classical realization of the *Lie-Santilli isoalgebras* [11-14,21]. Recall that the conventional classical realization of the Lie product is given by the familiar Poisson brackets among two functions A(b) and B(b) in the phase space,

$$[A, B]_{Poisson} = \frac{\partial A}{\partial r^k}\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial r^k}\frac{\partial A}{\partial p_k} = \frac{\partial A}{\partial b^\mu}\,\omega^{\mu\nu}\,\frac{\partial B}{\partial b^\nu}. \qquad (2.3.60)$$

The Birkhoffian mechanics characterizes instead the following brackets which are still Lie, yet of their most general possible type in conventional spaces over conventional fields [5]

$$[A, B]_{Birkh.} = \frac{\partial A}{\partial b^\mu}\,\Omega^{\mu\nu}(b)\,\frac{\partial B}{\partial b^\nu}, \quad \Omega^{\mu\nu} = [\,(\,\Omega_{\alpha\beta}\,)^{-1}\,]^{\mu\nu}. \qquad (2.3.61)$$

The isohamiltonian mechanics characterizes instead the following isobrackets among isofunctions $\hat{A}(\hat{b})$, $\hat{B}(\hat{b})$ in isospace first introduced in ref.s [32-35]

$$[A, B]_{Isot.} = \frac{\hat{\partial} A}{\hat{\partial} \hat{r}^k}\frac{\hat{\partial} B}{\hat{\partial} \hat{p}_k} - \frac{\hat{\partial} B}{\hat{\partial} \hat{r}^k}\frac{\hat{\partial} A}{\hat{\partial} \hat{p}_k} =$$

$$= \frac{\partial A}{\partial \hat{r}^k}\frac{\partial B}{\partial \hat{p}_k} - \frac{\partial B}{\partial \hat{r}^k}\frac{\partial A}{\partial \hat{p}_k}, \qquad (2.3.62)$$

which can be written in conventional spaces (from the isotopic character of the metric in the contraction of the k-indices)

$$[A, B]_{Isot.} = \frac{\partial A}{\partial \hat{r}_i}\hat{T}_i^{\,k}(t, r, p, ...)\,\delta_{kj}\,\frac{\partial B}{\partial \hat{p}_j} - \frac{\partial B}{\partial \hat{r}_i}\,\hat{T}_i^{\,k}(t, r, p, ...)\,\delta_{kj}\,\frac{\partial A}{\partial \hat{p}_j} \qquad (2.3.63)$$

As a result, it is easy to see that isobrackets (3.662) verify the Lie axioms in isospace but not necessarily in conventional space. Also, one should keep in mind from the comments at the beginning of this section that we have selected the

mechanics and studied in detail in monographs [9,10].

In this section we study the isotopies of the simplest possible quantization, called *naive isoquantization*. The naive quantization is the map of the canonical action functional of Eq. (3.5)

$$A^\circ (t, r) \quad \to \quad -i \hbar \, Ln \, \psi(t, r), \tag{2.3.66}$$

where $\hbar = 1$ is the unit of quantum mechanics, which maps the conventional Hamilton–Jacobi equations into Schrödinger's equations for the energy and momentum,

$$i \, \partial\psi(t, r) / \partial t \; = \; H(t, r, p) \, \psi(t, r) \, ,$$

$$p_k \, \psi(t, r) \; = \; - \, i \, \partial\psi(t, r) / \partial r^k \, , \tag{2.3.66}$$

with canonical commutation rules in the unified notation

$$[\, b_\mu \, , \, b_\nu \,] \; = \; b_\mu \, b_\nu \; - \; b_\nu \, b_\mu \; = \; \omega_{\mu\nu} \; I \, , \qquad b \; = \; \{ \, x_k, \, p_k \, \} \tag{2.3.67}$$

which are at the foundation of quantum mechanics, as well known.

Since the generalized action $\hat{A}^\circ$ is an isotopy of A°, the preceding map must also subjected to an isotopy. We reach in this way the following *naive isoquantization* [32,34]

$$\hat{A}^\circ(\hat{t}, \hat{r}) \quad \to \quad - \, i \, \hat{1}(\hat{t}, r, \dot{r}, \ddot{r}, \; \psi, \partial\psi, \partial\partial\psi, \mu \, \tau, n, ...) \, Ln \, \hat{\psi}(\hat{t}, \hat{r}) \, , \quad \hbar \; = \; 1 \, , \tag{2.3.68}$$

where $\hat{1}$ is the nonlinear and nonlocal isounit of the isotopic Newton equations with the additional quantum mechanical dependence on the wavefunction and its derivatives. From the outset one can therefore see that hadronic mechanics represent nonlinear and nonlocal systems.

The application of map (3.68) to Eq.s (3.55) yields: the *isoschrödinger equation in the energy*

over $\hat{C}$

$$< \hat{\psi} \hat{|} \hat{\phi} > = < \hat{\psi} | \times \hat{T} | \hat{\psi} > \hat{1} = \hat{1} \times \int d\hat{x}^3 \hat{\psi}^{\hat{\dagger}} \times \hat{T} \times \hat{\phi} \in \hat{C}(\hat{c},+,\hat{\times}), \qquad (2.3.72)$$

The isotopies of the Schrödinger's representations have been identified in the preceding section. The isotopies of the Heisenberg representation are characterized by the *isoheisenberg equation* for an observable $\hat{O}$ (first introduced in memoir [6b] of 1978)

$$i \hat{\partial} \hat{O} / \hat{\partial} t = [\hat{O}, \hat{H}] = \hat{O} \hat{\times} \hat{H} - \hat{H} \hat{\times} \hat{O} = \hat{O} \hat{T} \hat{H} - \hat{H} \hat{T} \hat{O}, \qquad (2.3.73)$$

The operator image of the classical Lie–Santilli isobrackets is therefore given by

$$[\hat{A}, \hat{B}] = A \hat{\times} B - B \hat{\times} B = \hat{A} \hat{T} \hat{B} - \hat{B} \hat{T} \hat{A}, \qquad (2.3.74)$$

and constitute the operator realization of the Lie–Santilli isoalgebra [11–14,21].

The exponentiated form of Eq.s (3.73) yields the time evolution of isostates

$$\hat{\psi}' = \hat{U} \hat{\times} \hat{\psi} = \{ \hat{e}^{i \hat{H} t} \} \hat{\times} \hat{\psi} = \{ (e^{i H Tt}) \times \hat{1} \} \hat{\times} \hat{\psi} = \{ e^{i \hat{H} T t} \} \times \hat{\psi}, \qquad (2.3.75)$$

where we have used the *isoexponentiation* of App. A. This yields an operator realization of the Lie–Santilli isogroups with laws (expressed in terms of an arbitrary isoparameters $\hat{w}$)

$$\hat{U}(\hat{w}) \hat{\times} \hat{U}(\hat{w}') = \hat{U}(\hat{w} + \hat{w}'), \quad \hat{U}(\hat{w}) \hat{\times} \hat{U}(-\hat{w}) = \hat{U}(\hat{0}) = \hat{1}. \qquad (2.3.76)$$

It is evidently impossible to review here the two monographs [9,10] on hadronic mechanics. We can therefore limit ourselves to an indication of only some of the basic aspects.

As a general rule, hadronic mechanics preserves all abstract axioms of quantum mechanics by conception and construction,. In fact, quantum and hadronic mechanics coincide at the abstract level for all isotopies of Class I for

above isoexpectation values coincide with the corresponding *isoeigenvalues*

$$\hat{H} \hat{\times} |\hat{\psi}> = \hat{E} \hat{\times} |\hat{\psi}> = (E \times \hat{1}) \times \hat{T} \times |\hat{\psi}> = E \times |\hat{\psi}> , \hat{E} \in \hat{R} , E \in R., \quad (2.3.80)$$

exactly as in the conventional case. Thus, the "numbers" of the theory which are confronted with experimental verifications are ordinary.

Similarly, the linear character of quantum mechanics (referred to the algebra of operators on a Hilbert space) is preserved in its entirety under isotopies, of course, when properly expressed in isohilbert space over an isofield with all products isotopic. In fact, the transformations must necessarily be isotopic for consistency, e.g., be of the type

$$\hat{r} \to \hat{r}' = \hat{A} \hat{\times} \hat{x} = \hat{A} \times \hat{T} \times \hat{r}, \quad p \to \hat{p}' = \hat{A} \hat{\times} \hat{p} = \hat{A} \times \hat{T} \times p , \quad \hat{A} \in \hat{\xi} , \quad (2.3.81)$$

and, as such, they do indeed verify the condition of linearity in isospace called *isolinearity*

$$\hat{A} \hat{\times} (\hat{n} \hat{\times} \hat{x} + \hat{m} \hat{\times} \hat{y}) = \hat{n} \hat{\times} (\hat{A} \hat{\times} \hat{x}) + \hat{m} \hat{\times} (\hat{A} \hat{\times} \hat{y}) , \quad (2.3.82)$$

for all possible $\hat{n}, \hat{m} \in \hat{R}$, $\hat{A} \in \hat{\xi}$, while its projection in the original space is nonlinear, e.g.,

$$r' = \hat{A}T(r, p, \psi, ...) r, \quad p' = A(t, r, p, \psi, ...) p . \quad (2.3.83)$$

As a result, the theory of isooperators on the isohilbert space $\hat{\mathcal{H}}$ over $\hat{C}$ is also isolinear. The regaining of locality in isospace, called *isolocality,* is established by the fact that the theory is everywhere local except at the unit. Finally, the regaining of canonicity in isospace, called *isocanonicity,* is established by the fact that the isoaction $\hat{A}^\circ$ is of first-order and coincides with the canonical action A° at the abstract level. The reconstruction of unitarity for nonunitary transforms when written in isospace has been indicated earlier. The same

number, rather than an isonumber.

property (3.86) implies the preservation of the numerical value of the uncertainties under isotopies,

$$\Delta r\, \Delta p \geq \tfrac{1}{2} |< r, \hat{p}]|> = \tfrac{1}{2} \hbar . \tag{2.3.87}$$

namely, the uncertainties of the center–of–mass of a particle are the same in quantum and hadronic mechanics [10].

This establishes that *the nonlinear, nonlocal and nonhamiltonian effects represented by Hamiltonian mechanics are "internal" and cannot be detected via center–of–mass experiments.* Alternatively, the virtual totality of contemporary experiments is unsuited for the distinction between quantum and hadronic mechanics because they are notoriously valid for the center–of– mass.

The experimental verification of hadronic mechanics therefore requires *new* experiments specifically conceived for *measures under "external" nonlinear, nonlocal and nonhamiltonian interactions,* exactly as it was historical the case for the electromagnetic interactions.

The reader should finally be aware that the above similarities between quantum and hadronic mechanics refer to their *formal* structure, because the study of actual systems with quantum and hadronic mechanics yields substantially different results.

This occurrence can be first expressed by the fact that *the same Hamiltonian* H *has basically different eigenvalues in quantum and hadronic mechanics,* as transparent from the presence of the isotopic element in the expressions

$$H \times |\psi> = E_o \times |\psi> \quad \rightarrow \quad H \hat{\times} |\hat{\psi}> = H \times \hat{T} \times \hat{\psi}> = E \times \hat{\psi}>, \; E_o \neq E \tag{2.3.88}$$

In reality, this is the very reason for which hadronic mechanics was built by permitting quantitative applications which are simply not possible with quantum mechanics, such as: the first exact numerical representation of the total magnetic moment of few body nuclei which, as well known, has escaped quantum mechanics for about one century; or the quantitative representation of

By recalling that the isotopic element $\hat{T}$ can depend explicitly on time in a time reversal asymmetric way, it follows that hadronic mechanics is indeed an operator formalism verifying conditions 1)–5) of part I.

Note finally that, unlike the physical case [10], the above formulation of hadronic mechanics of Class III has the intrinsic possibility of both motion forward and backward in time all in a fully causal way.

2.3.G: Problematic aspects of other generalizations of quantum mechanics

The reader should be aware that, as studied in detail in [10], several other generalizations of quantum mechanics exist in the literature which are mathematically correct, yet possess numerous problematic aspects, insufficiencies or shear inconsistencies in applications.

As a first example, nonlinearity (usually referred to the wavefunctions in operator theories) is often realized via a dependence of the Hamiltonian on ψ according to the generalized equation

$$i\,\partial\psi(t, r) / \partial t = H(t, r, p, \psi) \times \psi(t, r)\,, \qquad (2.3.90)$$

The problem, is that the above theory violates the principle of superposition, as well known, thus rendering the theory without practical value for any composite system.

Note that systems (3.90) can be *identically* reformulated in the isotopic form

$$i\,\partial\psi / \partial\tau = H(t, r, p, \psi)\,\psi \equiv H_0(t, r, p) \times \hat{T}(\psi, ...) \times \psi\,, \qquad (2.3.91)$$

which resolves the above problematic aspects in full, as ensures by the isolinearity of the theory.

3: ELEMENTS OF GENOTOPIC METHODS

3.1: Statement of the problem

All isotopic methods outlines in the preceding analysis admit a simple, yet unique and significant *genotopic image,* that is, a formulation in which the hermiticity of the basic isounit is relaxed,

$$\hat{1}^{\dagger} \neq \hat{1}, \tag{3.1}$$

resulting in *two* different, n×n–dimensional, generalized units, one used for *motion forward in time* and the other for *motion backward in time,* which are indicated in the literature with the symbols

$$\hat{1}^{>}(t, r, \dot{r}, \ddot{r}, ...) = S^{-1}, \qquad \hat{1}^{<}(t, r, \dot{r}, \ddot{r}, ...) = R^{-1}, \tag{3.2}$$

For consistent applications, the two generalized units must have an interconnecting relation which is generally assumed to be Hermiticity,

$$\hat{1}^{>} = (\hat{1}^{<})^{\dagger}. \tag{3.3}$$

However, in most applications, the generalized units are given by *real valued n×n–dimensional nonsingular matrices,* in which case the above interconnection is characterized by transposed,

The main advance of the genotopic over the isotopic methods is an axiomatic formulation of irreversibility, that is, the representation of irreversibility via methods which are irreversible irrespective of the time-reversal character of the individual elements, such as the Hamiltonian.

In particular, genotopic formulations permit a deeper understanding of time, as a quantity possessing in general *four* rather than the conventional two realizations.

While isotopic formulations are naturally set to represent total conservation laws under nonconservative internal effects, genotopic formulations generally have no conserved quantity, because they have been conceived to characterize *time-rate-of-variations of any given quantity* of which conservation is an evident particular case. Genotopies are therefore ideally suited to represent the nonconservative growth of biological structures as per condition 1) of Sect. I.2.

The genotopies were first discovered by Santilli in memoir [6b] of 1978 and then studied in monographs [4-5,7,8,9-10]. They are today generally referred to as *Santilli's genotopies* and are referred to a step-by-step generalization of the isotopic methods of Sect. II for nonhermitean units.

Needless to say, to avoid a prohibitive length, in this section we can only indicate the main lines of genotopic methods and refer to monographs [9,10) for details.

3.2: Mathematical foundations

The genotopies are first classified into *Kadeisvili's Classes I, II, III, IV, and V* [25], although now the conditions for each class is referred to the Hermitean component of the genounits. In particular, the genotopies particularly important for theoretical biology, those of Class III assumed hereon, are those for which the Hermitean component of the isounits can be either positive or negative.

The understanding of genotopic methods requires the knowledge that they are based on generalized numbers, called *genonumbers,* which are broader than

right, yielding the *forward genofield of Class I*

$$\hat{F}^{>}(\hat{a}^{>},+,\hat{\times}^{>}), \qquad \hat{a}^{>} = a \times \hat{1}^{>}, \tag{3.8}$$

whose elements $\hat{a}^{>}$ are called *forward genonumbers*, and one to the left, yielding the *backward genofield of Class I*

$$^{<}\hat{F}(^{<}\hat{a},+,^{<}\hat{\times}), \qquad ^{<}\hat{a} = {}^{<}\hat{1} \times a, \tag{3.9}$$

whose elements $^{<}\hat{a}$ are called *backward genonumbers*, plus the corresponding *isoduals genofields of Class II*

$$\hat{F}^{>d}(\hat{a}^{>d},+,\hat{\times}^{>d}), \qquad \hat{a}^{>d} = a \times \hat{1}^{>d}, \ \hat{1}^{>d} = -\hat{1}^{>}, \tag{3.10a}$$

$$^{<}\hat{F}^{d}(^{<}\hat{a}^{d},+,^{<}\times^{d}), \qquad ^{<}\hat{a}^{d} = {}^{<}\hat{1}^{d} \times a, \ {}^{<}\hat{1}^{d} = - {}^{<}\hat{1} . \tag{3.10b}$$

The above genofields are often denoted with the unified symbol $^{<}\hat{F}^{>}(^{<}\hat{a}^{>},+,^{<}*^{>})$ and $^{<}\hat{F}^{>d}(^{<}\hat{a}^{>d},+,^{<}\hat{\times}^{>d})$ with the understanding that the orderings can solely be used individually.

The reader should finally keep in mind that the *genofields of Class III admit as particular cases those those of Classes I and II.*

Note the *necessity* of an isotopy $a\beta \to a\hat{1}\beta$ in order to construct the genotopies. In fact, the ordering for conventional fields has no practical value.

As it will be evident in the applications, *the product ordered to the right can be interpreted as characterizing motion forward in time, while that ordered to the left can represent motion backward in time.* In different term, the ordering of Definition III.1 can represent *Eddington's "arrows of time"*, and we have the following:

Lemma 3.1 [9]: *An axiomatization of irreversibility in number theory is given by: A) the ordering of the multiplications to the right and to the left, representing motion forward and backward in time, respectively; B) the*

$$\hat{E}^{>}(\hat{r}^{>},\hat{\delta}^{>},\hat{R}^{>}) : \hat{g}^{>} = g \times S, \hat{r^{2}}^{>} = (r^{t} \times \hat{\delta}^{>} \times r) \times \hat{1}^{>} \in \hat{R}^{>}, \hat{1}^{>} = S^{-1}, \quad (3.13a)$$

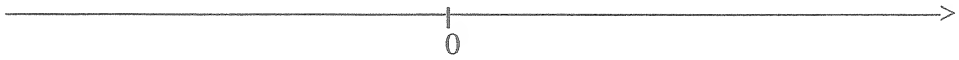
$$^{<}\hat{E}(^{<}\hat{r},^{<}\hat{g},^{<}\hat{R}) : {}^{<}\hat{G} = R \times g, r^{<2} = {}^{<}\hat{1} \times (r \times {}^{<}\hat{\delta} \times r^{t}) \in {}^{<}\hat{R}, {}^{<}\hat{1} = R^{-1}. \quad (3.13b)$$

$$\hat{1}^{>} = ({}^{<}\hat{1})^{t}, \quad (3.13c)$$

under which the simplest possible realization of genounits and genotopic elements is that via *3×3 real-valued, nonsymmetric matrices.*

GENOTIME

Motion forward in past time $t^{>d}$; Motion forward in future time $t^{>}$



Motion backward in past time $^{<}t^{d}$; Motion backward in future time $t^{>d}$

FIGURE 3.1: A schematic view of the four possible "time arrows" which are characterized by genotopic methods via four possible genounits interconnected by complex conjugation and isoduality. Note that the isotime and its isodual are a particular case when the forward and backward genounits coincide. As indicated in the application a generalization of the current notion of time appears to be necessary for quantitative interpretation of a number of biological aspects.

The *genoeuclidean geometry* is the geometry of the genoeuclidean spaces and it is given by a step-by-step lifting of the isoeuclidean geometry of Sect. II.2 whose explicit study is omitted here for brevity.

The genospaces and genogeometries were introduced for the first time by Santilli (see the general study in monographs [7-10]).

A most visible difference between genospaces and isospaces is therefore that the interval in the former is unique, while in the latter we have *four* different intervals, one for the motion forward and one for motion backward in time plus their isoduals which are all contained as particular cases of Class III.

genodifferential calculus, here introduced apparently for the first time. It is based on the *forward and backward genodifferentials*

$$\hat{d}^{>}r^{k} = \hat{1}^{>k}_{i}\, dr^{i}\,, \quad {}^{<}\hat{d}r^{k} = {}^{<}\hat{1}^{k}_{i}\, dr^{i}\,, \tag{3.14}$$

with corresponding *forward and backward genoderivatives*

$$\hat{\partial}^{>} / \hat{\partial} r^{k>} = S_{k}^{\ i}\, \partial / \partial r^{i}\,, \quad {}^{<}\hat{\partial} / {}^{<}\partial r^{k} = R_{k}^{\ i}\, \partial / \partial r^{i}\,. \tag{3.15}$$

which must be used, for consistency, only individually per each time ordering and with the corresponding ordered multiplications.

The genotopies of the remaining aspects can be worked out by the interested reader via similar methods.

3.3: Classical and operator genotopic dynamical equations.

We close this brief outline with the identification of the basic genotopic dynamical methods and their proper formulations.

The most fundamental dynamical equations of this section are the *genonewton equations,* here introduced apparently for the first time, which can be written for motion forward in time

$$\hat{m}^{>}\, \frac{\hat{d}^{>}\, \hat{v}^{>}_{k}}{\hat{d}t^{>}} - F_{k}^{>}(t^{>}, \hat{r}^{>}, \hat{v}^{>})\,, \quad k = x, y, z\,,$$

$$\hat{v}^{>}_{k} = \hat{d}^{>}\, \hat{r}^{>}_{k} / \hat{d}t^{>} = \hat{d}^{>}\, \hat{\delta}^{>}_{k}{}^{i} / \hat{d}t^{>}\,, \tag{3.16}$$

are the components of the *forward velocity.*

For consistency, the above equations must be computed over the forward genospace

$$\hat{S}^{>}(t^{>}, \hat{r}^{>}, \hat{v}^{>}) = \hat{E}^{>}(t^{>}, \hat{R}_{t}^{>}) \times \hat{E}^{>}(\hat{r}^{>}, \hat{\delta}^{>}, \hat{R}^{>}) \times \hat{E}(\hat{v}^{>}, \hat{\delta}^{>}, \hat{R}^{>})\,, \tag{3.17}$$

where we assume for simplicity $\hat{1}^{>}_{p} \equiv (\hat{1}^{>}_{r})^{-1}$, as suggested by the contravariant character of $\hat{r}^{>k}$ and the covariant character of $\hat{p}^{>}_{k}$.

It is easy to see that, under the necessary conditions for invertibility, the preceding theory can be equivalently derived via the *genolegendre transforms* into the first-order *forward genocanonical genoaction*

$$\hat{A}^{>} = \int_{t_1}^{t_2} [\, \hat{p}^{>}_{k}\, \hat{d}\hat{r}^{>k} - \hat{H}^{>}(t^{>}, \hat{r}^{>}, \hat{v}^{>})\, \hat{d}t^{>} \,] =$$

$$= \int_{t_1}^{t_2} [\, \hat{R}^{>}_{\mu}\, \hat{d}\hat{b}^{>\mu} - \hat{H}^{>}(t^{>}, \hat{r}^{>}, \hat{v}^{>})\, \hat{d}t^{>} \,], \qquad (3.23a)$$

$$\hat{R}^{>} = (\hat{p}^{>}, o), \quad \hat{b}^{>} = (\hat{r}^{>}, \hat{p}^{>}), \qquad (3.23b)$$

with ensuing *forward genohamilton equations*, here introduced apparently for the first time,

$$\frac{\hat{d}^{>}\hat{r}^{>k}}{\hat{d}t^{>}} = \frac{\partial^{>}\hat{H}^{>}}{\partial\hat{p}_{k}^{>}}, \qquad \frac{\hat{d}^{>}\hat{p}_{k}^{>}}{\hat{d}t^{>}} = -\frac{\partial^{>}\hat{H}^{>}}{\partial\hat{r}^{>k}}, \qquad (3.24)$$

or in unified notation

$$\left(\frac{\partial^{>}\hat{R}^{>}_{\nu}}{\partial\hat{b}^{\mu>}} - \frac{\partial^{>}\hat{R}^{>}_{\mu}}{\partial\hat{b}^{\nu>}}\right) \frac{\hat{d}^{>}\hat{b}^{\nu}}{\hat{d}t^{>}} - \frac{\partial^{>}\hat{H}^{>}}{\partial\hat{b}^{\mu>}}^{A} = 0, \qquad (3.25)$$

The above equations can be simply written in the following covariant and contravariant forms, respectively,

$$\omega_{\mu\nu}\, \frac{\hat{d}^{>}\hat{b}^{\nu}}{\hat{d}t^{>}} = \frac{\partial^{>}\hat{H}^{>}(t^{>}, \hat{b}^{>})}{\partial^{>}\hat{b}^{\mu>}}, \qquad (3.26a)$$

$$\frac{\hat{d}^{>}\hat{b}^{\mu>}}{\hat{d}t^{>}} = \omega^{\mu\nu}\, \frac{\partial^{>}\hat{H}^{>}(t^{>}, \hat{b}^{>})}{\partial\hat{b}^{\nu>}}, \qquad (3.27b)$$

where the quantities

$$\hat{p}^{>}_{k} > \hat{\psi}^{>}(\hat{t}^{>}, \hat{r}^{>}) = - i \frac{\hat{\partial}^{>} \hat{\psi}^{>}(\hat{t}^{>}, \hat{r}^{>})}{\hat{\partial} \hat{r}^{k}} = - i R_{k}{}^{i} \frac{\partial \hat{\psi}^{>}}{\partial r^{i}}. \qquad (3.31b)$$

The corresponding genotopies of Heisenberg equations are given by the following *genoheisenberg equations* introduced for the first time by Santilli in memoir [6b) of 1978

$$i \hat{\partial}^{>} \hat{Q} / \hat{\partial} \hat{t}^{>} = (\hat{Q}, \hat{H}) = Q < \hat{H} - \hat{H} > \hat{Q} =$$

$$= \hat{Q} \times R(t, r, p, \dot{p}, \psi, \partial\psi, \partial\partial\psi, \ldots) \times \hat{H} - H \times S(t, r, p, \dot{p}, \psi, \partial\psi, \partial\partial\psi, \ldots) \times Q, \qquad (3.32)$$

whose brackets characterize a *Lie–Santilli genoalgebra,* where we have assumed a knowledge by the reader on the different character of the time evolution in the transition from the Schrödinger to the Heisenberg representation.

The exponentiated form is given by

$$\hat{Q}(t) = \{ e^{iHRt} \} \times Q(0) \times \{ \exp^{-itRH} \}, \qquad (3.33)$$

which characterize a *Lie–Santilli genogroup* (see [9,10] for their full expression in terms of genoproducts).

The fundamental algebraic brackets of the theory

$$(A, B) = A < B - B > A, \qquad (3.34)$$

are at the foundations of the genotopies and characterize a *Lie–admissible algebra* according to the original identification by Albert [44] and the subsequent definitions by Santilli [6]. [9]

[9] Albert [44] defined a *Lie–admissible algebra* as a nonassociative algebra U with elements a, b, ... and abstract product ab such that the attached algebra U^{-} which has the same elements a, b, c, and the new product $[a, b]_U = ab - ba$, is Lie. This definition however was proved to be insufficient for applications because such an algebra U does not necessarily contain a Lie algebra as a particular case.

$$i\,dH/dt = [H, H] = H \times H - H \times H \equiv 0, \tag{3.36}$$

as well as for the isotopic theory,

$$i\hat{d}\hat{H}/\hat{d}t = [\hat{H}, \hat{H}] = \hat{H} \times \hat{T} \times \hat{H} - \hat{H} \times \hat{T} \times \hat{H} \equiv 0, \tag{3.37}$$

while the same Hamiltonian is not conserved for the genotopic theory by conception and construction.

$$i\,dH / dt = (H, H) = H(R - S)H \neq 0. \tag{3.38}$$

A simple example can be instructive here. Consider a free quantum mechanical particle with Hamiltonian $H_o = \frac{1}{2} p_o^2$, m = 1, which is evidently Hermitean over $\mathcal{C}$. Suppose now that this particle at a given instant of time t_o enters within a resistive medium, thus losing energy to the medium itself. Assume the simplest possible decay, the linearly damped one

$$H = e^{-\gamma t} H_o = e^{-\gamma t} \tfrac{1}{2} p_o^2. \tag{3.39}$$

Then a genotopic representation of the above system is given by

$$R = -\tfrac{1}{2} i \gamma H_o^{-1}, \quad S = +i \tfrac{1}{2} \gamma H_o^{-1}, \quad R = S^\dagger, \tag{3.40}$$

The Lie-admissible group of the time evolution of a quantity Q is then given by

$$Q(t) = \{ e_{\hat{\xi}>}^{i H (t_o - t)} \} > Q(t_o) < \{ e_{<\hat{\xi}}^{i (t - t_o) H} \} =$$

$$= e^{i H_o S (t_o - t)} Q(t_o) e^{-i (t_o - t) R H}. \tag{3.41}$$

with general infinitesimal Lie-admissible equation

$$i\,dQ / dt = Q < H_o - H_o > Q \tag{3.42}$$

(e.g., the Cooper pair) and repulsive in triplet coupling.

PHASE III: Noncanonicity. The effects due to mutual penetration of the wavepackets are beyond the descriptive capability of a Hamiltonian. As such, they occur without any need of energy, thus carrying a variety of implications beyond any physical description (Sect. 5.1).

The above main lines are sufficient for the limited objectives of this monograph. More detailed studies and applications, e.g., to molecular chains, are left to the interested biologist.

5.4. Apparent isotopic origin of correlations

Another important aspect where generalized methods are expected to provide a significant contribution is the quantitative representation of the *correlation* of particles, here generically referred to *any given relationship among quantum mechanical or classical particles at large mutual distances without any conventional interaction derivable from a potential, e.g., of electromagnetic type.*

This is another area of theoretical biology in which isotopic or more general methods can provide a rather unique contribution. Because of extended use, we have been accustomed throughout this century to restrict all possible interactions among particles to have *potential energy*, that is, admitting of a potential which is added to a Lagrangian or a Hamiltonian. Whether classical or quantum mechanical, available methods are then restricted to interactions of the latter type.

Correlations are instead due to interactions which should be represented with *anything except a potential or a Lagrangian or a Hamiltonian* because they do not carry potential energy. As such, they are conceptually and technical outside the realistic representational capabilities of conventional methods.

The isotopic, genotopic or hyperstructural methods are ideally suited for the problem considered because *correlations can be represented with the generalized units, without any need of a potential, or a Lagrangian, or a Hamiltonian.*

We can therefore introduce in theoretical biology yet another novel notion: interactions among particles which carry no energy of any kind, yet which produce

has been conceived and constructed precisely for the study of nonlocal interactions (as illustrated in the preceding subsections).

Moreover, protons and antiprotons are not ideal spheres with points in them, but are instead constituted by some of the densest media measured in laboratory by mankind until now. Being the result of the mutual penetration of the *hyperdense* protons and antiprotons, the fireball is therefore one of the most general known non–local integral systems. But *nonlocal–integral interactions are nonhamiltonian both conceptually and technically.* It then follows again that quantum mechanics is not expected to be exactly valid because of its central requirement of representing everything with one single quantity, the Hamiltonian.

But even assuming that the above transparent and basic insufficiencies are by–passes via not so infrequent machinations to preserve old knowledge (e.g., the addition the a Hamiltonian of the "nonlocal–integral potential" which, as pointed out in Sect. 2, has no mathematical or physical sense), quantum mechanics still remains structurally unable to represent correlation in an *exact* way.

This is due to the limitations of its very axioms as compared to the experimental evidence of the correlation. As an example, the two–body quantum mechanical *axiom of expectation value* is given by

$$C_n = \overbrace{\langle 1,a \,|\, \langle 1,b \,|\, \ldots\ldots \langle n,a \,|\, \langle n,b \,|}^{\left(\begin{array}{l} |\, 1,a\rangle \\ |\, 1,b\rangle \\ \ldots\ldots \\ |\, n,a\rangle \\ |\, n,b\rangle \end{array} \right)} =$$

$$= \sum_k \left(\langle k,a \,|\, k,a\rangle + \langle k,b \,|\, k,b\rangle \right), \qquad (5.58)$$

The above expression lacks exactly the cross terms $< k, a \,|\, k, b >$ representing the correlation. By comparison, the axiom of *isoexpectation values* of hadronic mechanics is given by (Sect. 2.3.F)

whenever needed.

Note that all quantities appearing in the isotopic element have a direct physical significance. In fact, S represents the shape of the fireball, and it is *numerically* determined by the experimental data (see below), $\mathcal{F} = 1$ for the considered BE correlation, the K–quantities are normalized to 1, and the off–diagonal terms (which are necessary to have correlation) represent precisely the nonlinear, nonlocal–nonhamiltonian interactions at the origin of the correlation itself, and are due to the mutual wave–overlapping of the wavepackets of the proton and antiproton. Note that when this overlapping is ignorable, the isotopic element is diagonal and there is no correlation, as expected and desired.

Under the *sole* approximation of that the longitudinal momentum transfer q_l is ignorable, as experimentally established and used in all data elaborations [47], relativistic hadronic mechanics then uniquely characterizes the correlation for two mesons expressible via the *two–points isocorrelation function*

$$\hat{C}_2 = 1 + (N^2 / 3) \sum_{\mu = 1, 2, 3, 4,} b_\mu^2 \eta_{\mu\mu} e^{-q_t^2 / b_\mu^2}, \qquad (5.62)$$

where q_t is the momentum transfer, $\eta = \text{diag} (\eta_{\mu\mu}) = \text{diag.} (1, 1, 1, -1)$ is the conventional Minkowski metric, $\hat{\eta} = \hat{S}\eta = \text{diag.} (b_1^2, b_2^2, b_3^2, - b_4^2)$ is the *isominkowski metric,* and $N^2 = b_1^2 + b_2^2 + b_3^2$ is a factor normalized to 3 (which is the sum of the semiaxes of the unit sphere).

By comparison, the semiphenomenological quantum mechanical models currently used are of the type

$$C_2 = 1 + \sum_k \lambda_k e^{-q_t^2 / \mu_k^2}, \qquad (5.63)$$

where λ_k and μ_k are parameters introduced *ad hoc* without any indication of their origin. The point is that the latter parameters are *prohibited* by the rigorous use of the quantum mechanical expectation value.

Memoir [48] concludes with the outline of the following results permitted by relativistic hadronic mechanics, *none of which is permitted by quantum mechanics*

formulated in a scale invariant way)

$$b_1^2 = 0.267 \pm 0.054, \quad b_2^2 = 0.437 \pm 0.035, \quad b_3^2 = 1.661 \pm 0.023, \quad b_4^2 = 1.653 \pm 0.015$$

thus yielding a very prolate spheroidal ellipsoids. This author would like to indicate that the above experimental verification of a central aspect of relativistic hadronic mechanics resulted to be beyond his most optimistic expectations.

isocorrelation function

$$C_2^{Max} = \lim_{q_t \to \infty} C_2 = 1 + 1/3 + 1/3 + 1/3 - 1/3 = 1.67\,, \quad \hat{C}_2^{Min} = 1\,, \tag{5.64}$$

which have resulted to be verified by experiments.

4) Prediction of the maximal possible value of the parameter b_4^2 characterizing the density of the fireball

$$1 + N^2 / 3 - N^2 b_4^2 = 1.67, \qquad b_4^2 = 2.33\,, \tag{5.65}$$

which also resulted to be verified by independent experiment (on the behaviour of the meanlives of unstable particles with speed).

6) Reconstruction of the exact Poincaré symmetry at the isotopic element for the above correlation *under nonlinear, nonlocal and nonhamiltonian interactions*.[17]

7) Exact derivation of the two-point isocorrelation function from first axiomatic principles without the introduction of *ad hoc* parameters, in a way remarkably in agreement with experimental data (Fig. 5.5)

The application of the above isotopic representation of correlation opens new intriguing possibilities in theoretical biology contemplated for study in subsequent papers because it is expected to provide a quantitative representation of effects whose cause was essentially unknown.

[17] The interested reader can verify the exact character of the isorotational symmetry $\hat{O}(3) \approx O(3)$ outlined in App. C for the space component of the fireball. The extension to the space-time setting including translations is consequential, *provided* that it is done in isominkowskian space over the isoreal isofields.

Class III appear to be particularly attractive on methodological grounds. In fact, the studies deal with *nonlinear* kinetic interpretations of biomolecular processes.[18] As such, their isotopic lifting is expected to reproduce linearity in isospace, while permitting the natural inclusion of nonlocal and nonhamiltonian effects. Moreover, the inclusion of bifurcations is expected to require the use of the isotopies of Class III with an isounit of time which can be positive or negative. Possible verifications of the lifting with experimental data would then raise the intriguing issues touched in Sect. 5.1 regarding a conceivable difference between our perception of time and the notion of time which may occur in reality in biomolecular processes.

Lifting of Dreismann-Streffer-Larhammar studies in [50]. The lifting of the latter studies indicates additional possibilities which may be worth an inspection. In fact, the studies include long range correlation and can therefore be an ideal test for the *isotopic correlation without energy* indicated in the preceding subsection. Moreover, the study address one of the most fundamental problems of contemporary theoretical biology, the DNA code. Their hyperstructural lifting is therefore equally intriguing on various grounds. We assume the reader is familiar with the fact that fractals are a particular case of the isotopies of Class V, and that the methods outlined in this memoir imply a sequential, rather intriguing, structural generalization of cryptography into the *iso–, geno– and hypercryptography* (see ref. [10], Ch. 2), the latter being the lifting of conventional cryptography (which, as well known, is based on the simple unit +1 dating back to biblical times), not only with infinitely possible units, but each generalized unit being multidimensional. In turn, such a study can illustrate the complexity of the DNA code indicated since the Abstract.

Lifting of Streffer-Hubner-Dreismann in [50]. These studies deal with correlation effects in the double proton transfer in DNA and, as such, are quite promising for a study of the possible isotopic or genotopic origin of the correlation itself, depending on the desired emphasis on reversibility or irreversibility.

[18] It is important to recall here that nonlinearity is sufficient alone to suggests brooder more adequate methods capable of preserving the superposition and other principles needed for composite systems (Sect. 2.3.G).

APPENDIX A: ELEMENTARY ISOFUNCTIONS

A.1: Foreword

We indicated in the preceding sections that the notion of angles, the conventional Pythagorean theorem, the trigonometric and hyperbolic functions and other familiar functions are inapplicable under isotopies for numerous independent reasons, such as: the loss of the conventional unit 1 in favor of generalized isounits $\hat{1}$; the inapplicability of the Euclidean distance; the generally curved character of the lines which prohibit the preservation of conventional angles; etc.

In this appendix we study the rudiments of the liftings of the Pythagorean theorem, trigonometric and hyperbolic functions which are applicable under isotopies. The *Isopythagorean Theorem, isotrigonometric* and *isohyperbolic functions* were identified by Santilli (see, e.g., note [44]) and presented in details in Appendix 5.C, Vol. I, ref. [9], to which we refer for individual contributions.

The above generalizations are evidently necessary for any calculation or application to biological systems, by keeping in mind that the use of *conventional* functions under *isotopies* leads to numerous inconsistencies.

In order to render this monograph selfsufficient, it appears recommendable to outline the methods for the construction of simple isofunctions, and leave to the interested reader the construction of the corresponding geno– and hyper–functions. We shall use the symbols $\hat{x}$, $\hat{A}$, $\hat{D}$, etc. to denote quantities computed in isospace and x, A, D, etc., to denote their projection in the original space.

their derivatives of arbitrary order $\hat{1} = \hat{1}(t, r, \dot{r}, \ddot{r}, ...)$.

The realization of $\hat{E}$ studied in this appendix is the simplest possible one of Class I, that with diagonal isounit, of the type

$$\hat{E}(\hat{r},\hat{\delta},\hat{R}) : \hat{r} = \{ \hat{r}^k \} = (\hat{x}, \hat{y}) \equiv \{ r^k \} = \{ x, y \}, \ \hat{r}_k = \hat{\delta}_{ki} \hat{r}^i \neq r_k = \delta_{ki} r^i, \quad (A.3a)$$

$$\hat{\delta} = \hat{T}(t, r, \dot{r}, \ddot{r}, ...) \delta = \text{diag.} (b_1^2 , b_2^2), \ b_k = b_k(t, r, \dot{r}, \ddot{r}, ...) > 0 , \quad (A.3b)$$

$$\hat{1} = \hat{T}^{-1} = \text{diag.} (b_1^{-2} , b_2^{-2}), \quad k = 1, 2 , \quad (A.3c)$$

The central notion of the isoeuclidean plane is the assumption of new (dimensionless) units, the quantities b_1^{-2} for the $\hat{x}$–axis and b_2^{-2} for the $\hat{y}$–axis. Thus, not only the unit is now different than +1, but different axes have different units and, in addition, each of them is a function of the local variables.

Consider now two points $\hat{P}_1(\hat{x}_1, \hat{y}_1)$, $\hat{P}_2(\hat{x}_2, \hat{y}_2) \in \hat{E}(\hat{r},\hat{\delta},\hat{R})$. Then the conventional distance is (uniquely) generalized into the *isoeuclidean distance* (Sect. 2.2.B)

$$\hat{D} = [(x_1 - x_2) b_1^2 (x_1 - x_2) + (y_1 - y_2) b_2^2 (y_1 - y_2)]^{1/2} \times \hat{1} \in \hat{R} , \quad (A.4)$$

where one should note the final (ordinary) multiplication by $\hat{1}$ as a necessary condition for $\hat{D}$ to be an element of the isofield $\hat{R}$.

Despite the visible difference between D and $\hat{D}$, all conventional notions in E are preserved under isotopies *provided* that they are computed in $\hat{E}$ over $\hat{R}$. In this way, we have the notions of *isolines, isostraight line, isotriangle, isostraight triangle,* etc. studied in Sect. 2. We then have the following:

Theorem A.1 (Isopythagorean theorem) [44]: *The following property holds in the isoeuclidean plane* $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ *of Class I,*

$$\hat{D}^{\hat{2}} = \hat{D} \hat{\times} \hat{D} = \hat{A}^{\hat{2}} + \hat{B}^{\hat{2}} = \hat{A} \hat{\times} \hat{A} + \hat{B} \hat{\times} \hat{B} \in \hat{R} , \quad (A.5)$$

the projection of Theorem 5.C.1 in the original Euclidean plane. Recall from Sect. I.5.2 that the isotopic lifting of the circle C in E yields the *isocircle* $\hat{C}$ in $\hat{E}$ which preserves the original geometric character including the value of the radius.

ISOPYTHAGOREAN THEOREM

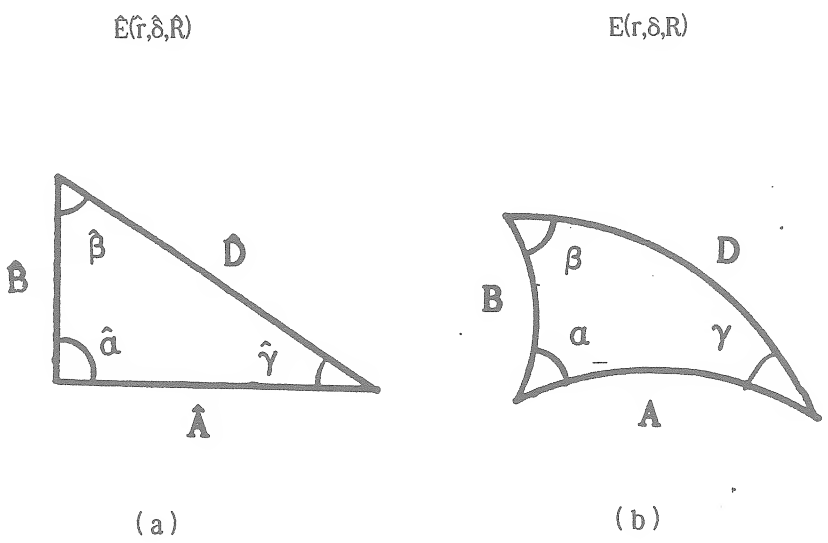


FIGURE A.1. A schematic view of the *Isopythagorean Theorem*, first identified in [44], for an *isoright isotriangle* as in Diag (a), i.e., a triangle in isoeuclidean plane $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ (isotriangle) with a 90° angle measured with respect to its own isounit (*isoright angle* – see below for its identification), and its projection in the conventional plane E(r,δ,R) given by the Diag. (b).

We also recall that isotopic maps are not transitive, in the sense that the lifting of the circle C on E into the isocircle $\hat{C}$ on $\hat{E}$ is axiom-preserving, but the projection of the isocircle $\hat{C}$ on the original space E is not, being in fact an ellipse, because such a projection does not imply the return to the original unit I = diag. (1, 1).

By using the reformulation in conventional space $\bar{E}$, it is easy to see that

$$\hat{D}^{\hat{2}} = (x\, b_1^2\, x + y\, b_2^2\, y) \times \hat{1} = \hat{1}\,, \quad \text{i.e.,} \tag{A.12a}$$

$$x\, b_1^2\, x + y\, b_2^2\, y = 1\,, \tag{A.12b}$$

which imply that for $y = 0$, $x = b_1^{-1}$ and for $x = 0$, $y = b_2^{-1}$.

By assuming the points $\hat{P}_1(\hat{x}_1, \hat{y}_1)$ and $\hat{P}_2(b_1^{-1}, 0)$, we have (for $0 < \hat{\alpha} < \pi/2$)

$$\cos \hat{\alpha} = x_1\, b_1\,, \tag{A.13}$$

and for the points $\hat{P}_1(\hat{x}_1, \hat{y}_1)$ and $\hat{P}_2(0, b_2^{-1})$ we have

$$\sin \hat{\alpha} = y_1\, b_2 \tag{A.14}$$

Definition A.1 [44]: *The "isosinus", "isocosinus" and other isotrigonometric functions on the isoeuclidean plane* $E(\hat{r},\hat{\delta},\hat{R})$ *are defined by (for* $0 < \hat{\alpha} < \pi/2$)

$$\text{isosin}\, \hat{\alpha} = b_2^{-1} \sin \hat{\alpha}\,, \tag{A.15a}$$

$$\text{isocos}\, \hat{\alpha} = b_1^{-1} \cos \hat{\alpha}\,, \tag{A.15b}$$

$$\text{Isotan}\, \hat{\alpha} = \frac{\text{isosin}\, \hat{\alpha}}{\text{isocos}\, \hat{\alpha}}\,, \tag{A.15d}$$

$$\text{Isocot}\, \hat{\alpha} = \frac{\text{isocos}\, \hat{\alpha}}{\text{isocos}\, \hat{\alpha}}\,, \tag{A.15e}$$

$$\text{isosec}\, \hat{\alpha} = 1/\, \text{isocos}\, \hat{\alpha}\,, \quad \text{isocosec}\, \hat{\alpha} = 1\, /\, \text{isosin}\, \hat{\alpha}\,. \tag{A.15f}$$

with basic property

$$\text{isocos}^{\hat{2}}\, \hat{\alpha} + \text{isosin}^{\hat{2}}\, \hat{\alpha} = b_1^2\, \text{isocos}^2\, \hat{\alpha} + b_2^{-2}\, \text{isosin}^{\hat{2}}\, \hat{\alpha} =$$

$$\hat{\alpha} = b_1(t, x, y, \dot{x}, \dot{y}, ...)\, b_2(t, x, y, \dot{x}, \dot{y}, ...)\, \alpha\,, \qquad (A.22)$$

but they have *constant values* in isospace because measured with respect to the angle isounit $\hat{1}_{\hat{\alpha}} = b_1^{-1}\, b_2^{-1}$. We reach in this way the following property:

Proposition A.1 [44]: *The isotopies of the plane geometry preserve the numerical value of the original angles, that is, if the original angle is* $\alpha = 90^\circ$ *so is the value of the corresponding isoangle* $\hat{\alpha}$ *is isospace.*

In fact, a given isotopic deformation of the angle $\alpha \to b_1 b_2 \alpha$ occurs under the joint *inverse* deformation of the basic unit $I \to \hat{1} = b_1^{-1} b_2^{-1}$, thus leaving the original numerical value α unchanged.

With respect to Fig. 5.C.1 we therefore have $\hat{\alpha} = 90^\circ$ and $\hat{\alpha} + \hat{\beta} + \hat{\gamma} = 180^\circ$. However, after the lifting $\alpha = 90^\circ \to \hat{\alpha} = 90^\circ$, the projection of the latter in the original plane does not yield back the angle $\alpha = 90^\circ$, but an angle α such that $\hat{\alpha} = b_1 b_2 \alpha = 90^\circ$ and similarly we have $\alpha + \beta + \gamma \neq 90^\circ$ but $\hat{\alpha} + \hat{\beta} + \hat{\gamma} = b_1 b_2 (\alpha + \beta + \gamma) = 180^\circ$. It is then easy to see that the isotrigonometric functions are periodic as in the conventional case, i.e.,

$$\text{isosin}\,(\hat{\alpha} + 2k\pi) \equiv \text{isosin}\,\hat{\alpha}\,, \qquad (A.23a)$$

$$\text{isocos}\,(\hat{\alpha} + 2k\pi) \equiv \text{isocos}\,\hat{\alpha}\,, \qquad k = 1, 2, 3, ... \qquad (A.23b)$$

and preserve the conventional symmetry under the inversion of the angles

$$\text{isocos}\,-\hat{\alpha} \equiv \text{isocos}\,\hat{\alpha}\,, \quad \text{isosin}\,-\hat{\alpha} = -\,\text{isosin}\,\hat{\alpha}\,. \qquad (A.24)$$

Similarly, we have the *Theorems of Isoaddition* [1]

$$\text{isosin}\,(\hat{\alpha} \pm \hat{\beta}) = b_1^{-1}\,(\text{isosin}\,\hat{\alpha}\,\text{isocos}\,\hat{\beta} \pm \text{isocos}\,\hat{\alpha}\,\text{isosin}\,\hat{\beta})\,, \qquad (A.25a)$$

$$\text{isocos}\,(\hat{\alpha} + \hat{\beta}) = b_1^{2}\,(b_2^{-2}\,\text{isocos}\,\hat{\alpha}\,\text{isocos}\,\hat{\beta} \pm b_1^{-2}\,\text{isosin}\,\hat{\alpha}\,\text{isosin}\,\hat{\beta}) \qquad (A.25b)$$

$$\hat{e}^{\,i\hat{a}} = \hat{1} + (i\hat{a})/1! + (i\hat{a})\hat{T}(i\hat{a})/2! + \ldots =$$

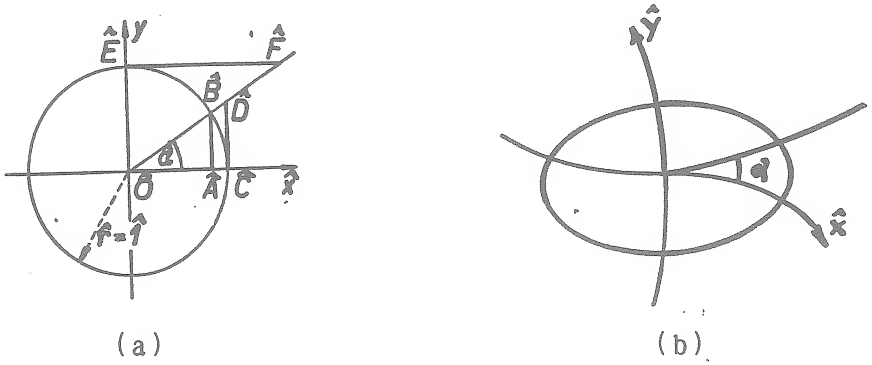
$$= \hat{1}_{\hat{a}} \times e^{i\hat{T}_{\hat{a}}\,a} = (b_1 b_2)^{-1} \times e^{i(b_1 b_2)a} =$$

$$= b_2^{-1}\,\text{isocos}\,\hat{a} + i\,b_1^{-1}\,\text{isosin}\,\hat{a}, \qquad (A.29)$$

where $\hat{e}$ denotes isoexponentiation and e conventional exponentiation.

ISOTRIGONOMETRIC FUNCTIONS ON THE ISOCIRCLE

$\hat{E}(\hat{r},\hat{\delta},\hat{R})$ $E(r,\delta,R)$



(a) (b)

FIGURE A.2: A schematic view of the isotrigonometric functions on the *isocircle* (Sect. I.5.2), that is, the circle in isospace, Diag. (a), and in its projection in conventional space, Diag. (b). Isotrigonometry shows that the the geometric structure of the circle is indeed axiomatic in the sense that it persists under isotopies. This is illustrated by the preservation under isotopy of the polar coordinates on the conventional circle (Diag. (a))

$$x = \cos a \quad \rightarrow \quad \hat{x} = \text{isocos}\,\hat{a}\,,$$

$$y = \sin a \quad \rightarrow \quad \hat{y} = \text{isosin}\,\hat{a}\,.$$

$$\hat{e}^{\alpha} = \hat{1}_{\hat{\alpha}}\, e^{\hat{1}\hat{\alpha}\alpha} = (b_1 b_2)^{-1} e^{(b_1 b_2)\alpha} =$$

$$= b_1^{-1} \text{ isocosh } \hat{\alpha} + b_2^{-1} \text{ isosinh } \hat{\alpha} . \qquad (A.32)$$

The interested reader can then work out the remaining properties of the isohyperbolic functions.

We now show the property that the distinction between trigonometric and hyperbolic functions is essentially due to the excessive simplicity of the basic unit customarily used in contemporary mathematics, while such a distinction is lost under more general units.

In fact, the use of a more general unit under isotopies allows the following result.

Lemma A.1 [44]: *Isotrigonometric and isohyperbolic functions lose any distinction on isoeuclidean planes* $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ *of Class III*

Proof. Assume the realization of the isounits $\hat{1}$ and $\hat{1}_{\hat{\alpha}}$ of Class III,

$$\hat{1} = \text{diag.}\,(g_{11}^{-1}, g_{22}^{-1}), \qquad \hat{1}_{\hat{\alpha}} = (g_{11}\, g_{22})^{-1/2}, \qquad (A.33)$$

where the functions $g_{kk} = g_{kk}(t, x, y, \dot{x}, \dot{y}, ...)$ are smooth, real-valued and nowhere null but otherwise arbitrarily positive or negative. Then, the isoexponential realization of the isotrigonometric functions (29) and of the isohyperbolic functions (32) are unified into the form

$$\hat{e}^{\hat{\alpha}} = \hat{1}_{\hat{\alpha}}\, e^{\hat{T}_{\hat{\alpha}}\alpha} = (g_{11}\, g_{22})^{-1/2}\, e^{(g_{11}\, g_{22})^{1/2}\alpha}, \qquad (A.34)$$

where the isotrigonometric functions occur when the product $g_{11}g_{22}$ is positive and the isohyperbolic functions occur when the same product is negative. **q.e.d.**

Lemma 1 also unifies the *conventional* trigonometric and hyperbolic functions, the former occurring for $\hat{1} = I$ = diag. (1, 1) or Dig. (−1, −1) and the latter for $\hat{1}$ = diag. (+1, −1) or Diag. (−1, +1), the second alternatives being the isodual of the first ones.

$$\hat{Q}(t) = \{ \hat{e}^{iHt} \} \hat{\times} Q(0) \hat{\times} \{ \hat{e}^{-itH} \} . \tag{A.38}$$

The *isologarithm* of an isonumber $\hat{a} \in \hat{F}(\hat{a},+,*)$ on isobasis $\hat{e} = e \hat{\times} \hat{1}$ can be defined as the quantity $\hat{log}_{\hat{e}} \hat{a}$ such that

$$\hat{e}^{\hat{log}_{\hat{e}} \hat{a}} = \hat{a} , \tag{A.39}$$

with evident (and unique) solution

$$\hat{log}_{\hat{e}} \hat{a} = \hat{1} \times \log_e a . \tag{A.40}$$

where $\log_e a$ is the ordinary logarithm on basis e of the ordinary number a. It is easy to see that the above definition characterizes a correct isotopy because it preserves all the conventional properties of log a, such as (we ignore in the following the subscripts $\hat{e}$ and e for simplicity)

$$\hat{log}\, \hat{e} = \hat{1}, \quad \hat{log}\, \hat{1} = 0, \tag{A.41a}$$

$$\hat{log}\, \hat{a} \hat{\times} \hat{b} = \hat{log}\, \hat{a} + \hat{log}\, \hat{b} , \quad \hat{log}\, \hat{a} \hat{/} \hat{b} = \hat{log}\, \hat{a} - \hat{log}\, \hat{b} , \tag{A.41b}$$

$$\hat{log}\, \hat{a}^{\hat{-1}} = - \hat{log}\, \hat{a} , \quad \hat{b} \hat{\times} \hat{log}\, \hat{a} = \hat{log}\, \hat{a}^{\hat{b}} , \text{ etc.} \tag{A.41c}$$

A similar situation occurs for the isotopy of most, but not all functions, including determinant, traces, matrices, etc. In fact, two exceptions are given by the isotopy of the trigonometric and hyperbolic functions studied earlier in this appendix.

For the isotopies of the remaining elementary and special functions we are forced to refer the reader to monographs [9,10]. The genotopies of elementary and special functions can be constructed accordingly, by just relaxing the hermiticity of the isounit, although they require special study because of the appearance of off-diagonal terms in the isounit under the condition of real-valuedness (or

APPENDIX B: ISOSPHERICAL COORDINATES

As it is well know, the polar and spherical coordinates are particularly useful in theoretical biology. It may then be recommendable to outline in this monograph for self–sufficiency their isotopic generalizations.

In ref. [10] we have proved that the conventional spherical coordinates in Euclidean space $E(r,\delta,R)$

$$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\theta, \tag{B.1}$$

with familiar measure

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2), \tag{B.2}$$

imply a number of inconsistencies when used in isospaces $\hat{E}(r,\hat{\delta},\hat{R})$, such as the impossibility of separating the radial and angular part in the equations of motion, and other problems, which persist in the use of other *conventional* coordinate systems, e.g., elliptical.

These occurrences have rendered mandatory the construction of the isotopies of the spherical coordinates, called isospherical coordinates, which are the correct coordinates for isospaces $\hat{E}(r,\hat{\delta},\hat{R})$.

We shall present first the simplest possible derivation of the isospherical coordinates, and then its more general form as needed for the isorepresentation theory of the isotopic $S\hat{O}(3)$ symmetry and other applications. Consider the three–

$$\hat{ds^2} = \bar{dx}\,\bar{dx} + \bar{dy}\,\bar{dy} + \bar{dz}\,\bar{dz} =$$

$$= dx\,T_x\,dx + dy\,T_y\,dy + dz\,T_z\,dz =$$

$$= dr^2 + r^2\,[\,T_z\,d\theta^2 + T_x\,T_y \sin^2\hat{\theta}\,d\phi^2\,] \equiv$$

$$\equiv dr^2 + r^2\,[\,d\hat{\theta}^2 + (\sin^2\hat{\theta})\,d\hat{\phi}^2\,]. \qquad \text{(B.8)}$$

The expression of the isomeasure for the general case in which the isotopic element depends on the local variables requires the full use of the differential calculus and, as such, it is omitted for brevity (see ref. [10]).

The isospherical coordinates in form (B.7) are useful for practical calculations, although they are not in their most general possible form because conventional trigonometric functions admit isotopic images. Their formulation in terms of the isotrigonometric functions then permits deeper insights.

Recall from Appendix A that the *isopolar coordinates* expressed in terms of the *isotrigonometric functions* in the *isogauss* (x, y)*-plane* with isotopic element $T = \text{diag.}\,(T_x, T_y)$ are given by

$$x = r\,\text{isocos}\,\hat{\phi} = r\,T_x^{-\frac{1}{2}} \cos[\,(T_x^{\frac{1}{2}}\,T_y^{\frac{1}{2}})\,\phi\,], \qquad \text{(B.9a)}$$

$$y = r\,\text{isosin}\,\hat{\phi} = r\,T_y^{-\frac{1}{2}} \sin[\,(T_x^{\frac{1}{2}}\,T_y^{\frac{1}{2}})\,\phi\,], \qquad \text{(B.9b)}$$

and verify the *isopythagorean theorem*

$$x\,T_x\,x + y\,T_y\,y = r^2\,(T_x\,\text{isocos}^2\,\hat{\phi} + T_y\,\text{isosin}^2\,\hat{\phi}) = r^2. \qquad \text{(B.10)}$$

In particular, the isotopic element of the above isotrigonometric functions is *not* that of the isogauss plane, but rather the element T in the isoexponentiation

$$\hat{e}_\phi^{\,i\phi} = \hat{1}_\phi\,e^{i\,T\,\phi} = \hat{1}_\phi\,e^{i\,\hat{t}\,\phi} = \hat{1}_\phi\,(\cos\hat{\phi} + i\sin\hat{\phi})\,] =$$

$$= r [B_{22}^{-1} \sin (B_{21} B_{22} \theta)] [B_{12}^{-1} \sin (B_{11} B_{12} \phi)] , \qquad \text{(B.14b)}$$

$$z = r \, \text{isocos} \, \hat{\theta} = r B_{21}^{-1} \cos (B_{21} B_{22} \theta) , \qquad \text{(B.14c)}$$

and isoidentity

$$x T_x x + y T_y y + z T_z z = \qquad \text{(B.15)}$$

$$= r^2 (B_{22}^2 B_{11}^2 \, \text{isosin}^2 \hat{\theta} \, \hat{\cos}^2 \hat{\phi} + B_{22}^2 B_{12}^2 \, \text{isosin}^2 \hat{\theta} \, \text{isosin}^2 \hat{\phi} +$$

$$+ B_{21}^2 \, \text{isocos}^2 \hat{\theta}) = r^2 .$$

ISOSPHERICAL COORDINATES

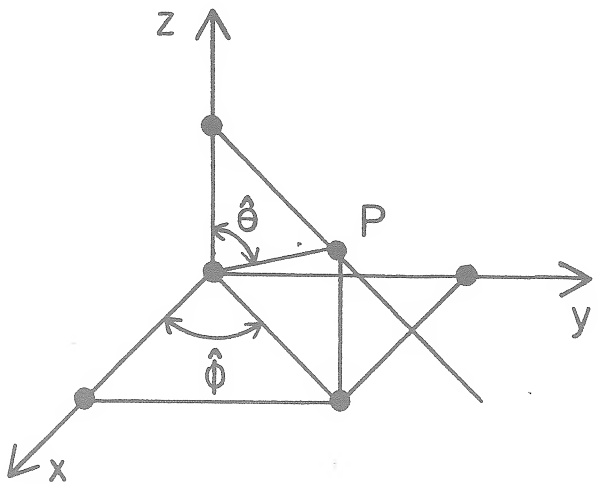


FIGURE B.1: A schematic view of the coordinates of a point on the isosphere in isoeuclidean space in three-dimension, Eq.s (B.14). As one can see, the representation coincides at the abstract level with the conventional one in Euclidean space with coordinates (B.1). However, the projection of the former in the space of the latter exhibits the ellipsoidical character of the isosphere.

Redefinitions (B.14) are important for the isorepresentation theory studied in ref. [5] because they permit the identification of the values of the *isospherical isotopic elements* and *isounits* separately for the θ and ϕ angles

APPENDIX C : ISOROTATIONAL SYMMETRY

As it is well known, the symmetry of the sphere in three-dimensional Euclidean space E(r,δ,R) is the *rotational symmetry* O(3). As it is equally well known, sea shells are believed not to admit a symmetry owing to their nonspherical shape as well as the increase of the shape itself in time.

In this appendix we outline the isotopies of the rotational symmetry which yield the invariance of arbitrary shapes when properly written in isospace. In turn, the availability of a basic symmetry for *nonconservative and irreversible* systems permits their reduction to primitive notions.

The *isorotational symmetry* $\hat{O}(3)$ was identified for the first time by this author [27] in 1985 (see the detailed studies in ref.s [10]). Such a symmetry is possible following the representation of sea shells in isoeuclidean spaces $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ and the identification of all possible shapes with the *isosphere* (Fig. 2.4).

Consider the isoeuclidean spaces $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ of Class III with isometric, isotopic element and isounit in the diagonal form (were we ignore hereon for simplicity the isoscalar character of $\hat{r} = r\hat{1}$)

$$\hat{\delta} = T\delta\,, \quad T = \text{diag.}\,(T_x, T_y, T_z), \quad T_k = T_k(t, r, \dot{r}, \ddot{r}, ...) > 0, \quad \hat{1} = T^{-1}. \tag{C.1}$$

The isotopies we are studying characterize the deformations of the sphere

$$r^2 = r^1 r^1 + r^2 r^2 + r^3 r^3 > 0, \tag{C.2}$$

the isomorphism $\hat{O}(3) \simeq O(3)$, *provided that it is realized at the covering isotopic level with respect to the isounit* $\hat{1} = T^{-1}$.

Note that the *conventional rotations* are indeed no longer a symmetry of the deformed sphere. Corollary C.1.A therefore focuses the attention on the difference between the violation of a *symmetry* in conventional spaces and its exact validity for the corresponding isospace. Equivalently, we are here referring to a mechanism of reconstruction of an exact symmetry in isospace when conventionally broken.

The isorotations can be explicitly written in $E(r,\delta,R)$

$$r' = \hat{R}(\theta) \hat{\times} r = \hat{R}(\theta)\, T(t, r, \dot{r}, \ddot{r}, ...)\, r = \tilde{S}(t, r, \dot{r}, \ddot{r}, ...)\, r\,, \quad \hat{R} = \tilde{S}\hat{1}\,, \qquad (C.4)$$

and therefore result to be intrinsically nonlinear. This is due to the fact that the functional dependence of the isotopic elements is completely unrestricted by the isotopies. We therefore have the following

Corollary C.1.B [loc. cit.]: *While conventional rotations are linear, local and canonical transformations in* $E(r,\delta,R)$, *isorotations are isolinear, isolocal and isocanonical in* $\hat{E}(r,\hat{\delta},\hat{R})$, *but nonlinear, nonlocal and noncanonical when projected into* $E(r,\delta,R)$

A further important result is the isotopic generalization of the conventional *Euler's theorem* on the general displacement of a rigid body with one point fixed which we can express via the following:

Theorem C.2 [loc. cit.]: *The general displacement of an elastic body with one fixed point is an isorotation* $\hat{O}(3)$ *of Class I around an axis through the fixed point.*

The above theorem illustrates the use of the classical isorotational symmetry for the characterization of deformable bodies.

A brief outline of the classical isorotational symmetry is the following. First,

showing the isotopy $I \to \hat{1}$. However, when considering the isocommutation rules between r_i and p_j we have

$$\begin{pmatrix} [r_i, r_j] & [r_i, p_j] \\ [p_i, r_j] & [p_i, p_j] \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{C.10}$$

Lemma C.1: *The classical isocommutation rules between* r_i *and* p_j *coincide with the conventional, canonical ones.*

Next, we introduce the generators of the Lie-isotopic algebra $\hat{so}(3)$ which, by central assumption, are given by the *conventional, contravariant generators* of O(3), [21]

$$J^k = \epsilon^{kij} r_i p_j = \epsilon^{kij} T_i r^i p_j . \tag{C.11}$$

The above quantities are called the components of the *isotopic angular momentum* to emphasize the fact that they characterize a generalized notion defined on $T^*\hat{E}(r,\hat{\delta},\hat{R})$ rather than on $T^*E_2(r,\delta,R)$.

In particular, the magnitude of the conventional angular momentum is given by the familiar expression $J^2 = J^k J_k = \delta^{ij} J_k J_k$, while the magnitude of the isotopic angular momentum is given by[22]

$$\hat{J}^2 = J * J = J^k * J_k = \hat{\delta}^{ij} J_i * J_j = \hat{\delta}^{ij} J_i T J_j . \tag{C.12}$$

Next, the following isocommutation rules are readily computed

$$[J^k, r_i] = \epsilon^{kij} r_j , \tag{C.13a}$$

[21] Unlike the operator case to be considered soon, note that the quantities r^i and p_j here are ordinary functions and, thus, they do not require the isotopic product $r^i * p_j$. Note also the subtle but important differences of the indices of $\hat{\delta} = (\hat{\delta}_{ij})$, $I = (\hat{1}^i{}_j)$ and $T = (T_i{}^j)$. Thus, only the tensor $\hat{\delta}_{ij}$ or its inverse $\hat{\delta}^{ij}$ used for lowering or raising indices.

[22] $\hat{J}^2$ is in this case an *isoscalar*, that is, a scalar quantity in isospace. For this reason it must be contracted in the form $\hat{J}^2 = \hat{J}^k * \hat{J}_k$.

isorotation around the third axis is given by [5]

$$r' = \hat{\mathcal{R}}(\hat{\theta}) * r = \tilde{S}(\hat{\theta})\, r = \tag{C.17}$$

$$= \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} x\cos\hat{\theta} - y\, T_x^{-\frac{1}{2}} T_y^{\frac{1}{2}} \sin\hat{\theta} \\ x\, T_x^{\frac{1}{2}} T_y^{-\frac{1}{2}} \sin\hat{\theta} + y\cos\hat{\theta} \\ z \end{pmatrix}.$$

$$\hat{\theta} = T_x^{\frac{1}{2}} T_y^{\frac{1}{2}}\, \theta\,. \tag{C.18}$$

It is an instructive exercise for the interested reader to prove the invariance of all possible deformed spheres under the above isotransforms.

We now compute a general isorotation which brings a point P on the isosphere to an arbitrary point Q. Its projection in Euclidean space is the transformation of a point P on an *ellipsoid* into another arbitrary point Q of the same ellipsoid. Such a rotation can be computed via three successive isorotations [10]:

1) An isorotation $\hat{\mathcal{R}}(\hat{\theta}_1)$ of an angle $\hat{\theta}_1 = T_x^{\frac{1}{2}} T_y^{\frac{1}{2}} \theta_1$ in the (x, y)–plane such that $0 \leq \hat{\theta}_1 \leq 2\pi$,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\hat{\theta}_1 & -T_x^{-\frac{1}{2}} T_y^{\frac{1}{2}} \sin\hat{\theta}_1 & 0 \\ T_x^{\frac{1}{2}} T_y^{-\frac{1}{2}} \sin\hat{\theta}_1 & \cos\hat{\theta}_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{C.19}$$

2) An isorotation $\hat{\mathcal{R}}(\hat{\theta}_2)$ around the polar axis z of an angle $\hat{\theta}_2 = T_z^{\frac{1}{2}} \theta_2$ such that $0 \leq \hat{\theta}_2 \leq \pi$,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\hat{\theta}_2 & -T_y^{\frac{1}{2}} T_z^{-\frac{1}{2}} b_3^{-1} \sin\hat{\theta}_2 \\ 0 & T_y^{-\frac{1}{2}} T_z^{\frac{1}{2}} \sin\hat{\theta}_2 & \cos\hat{\theta}_2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{C.20}$$

3) An isorotation $\hat{\mathcal{R}}(\phi_2)$ in the (x, y)–plane with angle $\hat{\theta}_3 = T_x^{\frac{1}{2}} T_y^{\frac{1}{2}} \theta_3$ such that $0 \leq \hat{\theta}_2 \leq 2\pi$,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\hat{\theta}_3 & -T_x^{-\frac{1}{2}} T_y^{\frac{1}{2}} \sin\hat{\theta}_3 & 0 \\ T_x^{\frac{1}{2}} T_y^{-\frac{1}{2}} \sin\hat{\theta}_3 & \cos\hat{\theta}_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. \tag{C.21}$$

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or the representation via the addition of an external collision term such as

$$i\, dQ / dt = Q \odot H = Q \times H - H \times Q + C, \quad H + H^{\dagger}, \tag{3.45}$$

The brackets $Q \odot H$ of the preceding equations *violate the conditions to characterize any algebra.* It then follows that the description in dissipative conditions, say, of "protons and neutrons with spin 1/2" has no mathematical or physical meaning of any nature, because of the absence in the theory of *all* algebras, let alone the SU(2) Lie algebra needed for the characterization of the spin 1/2 in a meaningful way.

Similarly, it is easy to prove that the time evolution of the latter dissipative equations is *nonunitary*, $U \times U^{\dagger} \neq I$. As a result, the preceding equations do not preserve their basic unit, $I \to I' = U \times I \times U^{\dagger} \neq I$ and, as such, the equations are not really applicable to experimental measures (which require an invariant unit, as well known). Moreover, the preceding equations are not invariant under their own time evolution, as one can readily verify. Therefore any interpretation or "number" derived via these theories has no known value of any type for applications. The reader may consult Sect. 7.2 of ref. [10] for additional problematic aspects or shear inconsistencies of these alternative formulations of dissipative systems.

In conclusion, to the best knowledge of this author at this time, the genotopic branch of hadronic mechanics presented in this section is the *sole* theory currently available for the description of dissipative or, more generally, nonconservative systems in a physically and mathematically consistent way.

In any case, alternative representation of dissipation are manifestly limited, while genotopic equations are directly universal for all possible nonlinear and nonlocal and nonhamiltonian nonconservative systems [9,10].

in hyperstructures from specific requirements in applications, apparently for the first time in the recent paper [16], that hyperstructures must possess a well defined right and left generalized unit (called *hyperunit*) to have meaningful applications. The related hyperstructures are then called $\hat{1}$*–hyperstructures.*

Ref. [16] also contains a mathematically rigorous presentation of $\hat{1}$–hyperstructures which we cannot possibly review here for brevity. This brief outline is intended to identify a specific realization of the $\hat{1}$–hyperstructures suitable for initial applications in theoretical biology, while we refer the mathematically inclined reader to paper [16].

Also, paper [16] presented the $\hat{1}$–hyperstructures as a generalization of the *isotopies* (i.e., under the condition of admitting the isotopies as particular cases), which is certainly correct and valid as a first approach. In this section we shall conduct a preliminary study of the broader $\hat{1}$–hyperstructures which admit the *genotopies* as particular case.

The latter requirement imposes additional conditions to select one given hyperstructure realized in explicit form, and leave the remaining possibilities as well as their abstract formulation for study after gaining sufficient input from applications.

We finally note that the $\hat{1}$–hyperstructures introduced in [16] are based on structural generalizations of conventional properties of associativity, commutativity, etc. called *weak associativity, weak commutativity,* etc. For reasons discussed in the applications, it appears advisable to study hyperstructures preserving the conventional properties and caklled *strong associativity,* etc.. As we shall see, this latter condition appears to be needed for consistency of the mathematical models with experimental observation.

In summary, in paper [16] we presented a particular class of hyperstructures which: 1) possess a well defined unit; 2) admit the isotopies as particular case; and 3) are defined in terms of weak operations. In this section we shall attempt the identification of hyperstructures which: 1) admit a well defined left or right unit; 2) admit the genotopies as a particular case; and 3) are defined in terms of strong operations.

It goes almost without saying that the latter hyperstructures will remain

elements is evidently important to avoid un–necessary difficulties.

Let $F(a,+,\times)$ be a conventional field of real, complex or quaternionic numbers with conventional sum $+$, multiplication $\times$, additive unit 0 and multiplicative unit 1. By following the same patterns as those for isotopies and genotopies, we define as *forward hyperfield* the set $\{\hat{F}^>\}(\{\hat{a}^>\}, + \{>\})$, where

$$\{\hat{a}^>\} = a \times \{\hat{1}^>\}, \qquad a \in F, \tag{4.2}$$

are the *forward hypernumbers* equipped with the conventional sum $+$ and additive unit 0 and the ordered multiplication to the right, called *forward hypermultiplication*

$$\{\hat{a}^>\}\{>\}\{\hat{\beta}^>\} = \{\hat{a}^>\} \times \{S_1, S_2, ...\} \times \{\hat{\beta}^>\}, \tag{4.3}$$

where all equalities are strong hereon, the quantity

$$\{S\} = \{S_1(t, r, \dot{r}, ...), S_2(t, r, \dot{r}, \ddot{r}, ...), ...\}, \tag{4.4}$$

called the *forward hyperelement,* is an ordered set of the same dimension of $\{\hat{1}^>\}$ and such that

$$\{\hat{1}^>\} \times \{R\} = \{1, 1, ...\}. \tag{4.5}$$

It is then easy to see that $\{\hat{1}^>\}$ is the correct left and right unit of $\{\hat{F}^>\}$

$$\{\hat{1}^>\}\{>\}\{\hat{a}^>\} \equiv \{\hat{a}^>\}\{>\}\{\hat{1}^>\} \equiv \{\hat{a}^>\}, \tag{4.6}$$

in view of the trivial property

$$\{1, 1, ...\} \times \{\hat{1}_1^>, \hat{1}_2^>, ...\} \equiv \{\hat{1}_1^>, \hat{1}_2^>, ...\}. \tag{4.7}$$

It follows that the hyperset $\{\hat{F}^>\}$ verifies all axioms to be a field in a strong

In fact, once explicitly worked out it reduces to the quantity Length × Unit × {1, 1, ... } and, in that sense, the above hypergeometries are axiom–preserving.

The *forward hyperdifferential calculus,* apparently introduced here for the first time, can be defined via the *forward hyperdifferentials*

$$\{ \hat{d}^{>} \}\, r^k = \{ \hat{1}^{>k}_{i} \} \times dr^i , \tag{4.10}$$

with corresponding *forward hyperderivatives*

$$\{\partial^{>}\} / \{ \hat{\partial} r^{k>} \} = \{ S_k{}^{i} \} \times \partial / \partial r^i , \tag{4.11}$$

with the corresponding ordered multiplications.

The hyperlifting of the remaining aspects can be worked out by the interested reader via similar methods.

4.3: Hyperstructural classical and quantum dynamical equations

The first and perhaps most important implication of the preceding section is that the above hyperstructures imply a further generalization of the four genotimes of Sect. 3.2 into *hypertimes of multivalued character for each of the four oriented directions of the genotimes,* according to the outline

Forward hypertime $\{t^{>}\} = t \times \{\hat{1}_t{}^{>}\}, \{\hat{1}_t{}^{>}\} = \{S_t{}^{-1}\}$,

Conjugated forward hypertime $\{^{<}t\} = t \times \{^{<}\hat{1}_t\}, \{^{<}\hat{1}_t\} = \{ R_t{}^{-1}\} = \{\hat{1}_t{}^{>}\}^{\dagger}$,

Backward hypertime $\{t^{>}\}^{d} = -\{t^{>}\}, \{\hat{1}^{>}\}^{d} = -\{\hat{1}^{>}\}$,

Conjugated backward hypertime $\{^{<}t\}^{d} = -\{^{<}t\}, \{^{<}\hat{1}\}^{d} = -\{^{<}\hat{1}\}$,

The map interconnecting motion forward with that backward (or viceversa) is called

$$\textit{time isoinversion} \qquad t^{>} \to {}^{<}t, \qquad \text{(b)}$$

and it is given by any bi–injective map suitable for the specific model at hand, e.g., complex conjugation, Hermitean conjugation, etc. The combination of the above two conjugation then yields the desired four different times, t, t^d, $t^{>}$, $(t^{>})^d$ one per each time arrow.

It is easy to see that the conventional time, here written $t\times 1$ where 1 stands for the assumed time unit (say, one sec), constitutes the simplest conceivable realization. Nevertheless, it is our *perception* of time and, as such, it does not necessarily correspond to an intrinsic reality. Moreover, the conventional time admits the isodual map $t^d = -t$, but it is not suitable to characterize the isoinversion. Finally, our conventional perception of time is structurally reversible and, as such, not suitable for a quantitative characterization of the irreversibility of biological processes.

In view of these insufficiencies, in this monograph we have introduced three generalizations of the conventional time for biological systems, called *isotime, genotime and hypertime*. They all share the property of verifying the same geometric axiom of our perception of time,

$$(\text{ period of time })^2 \times (\text{ unit }) = \text{ invariant }, \qquad \text{(c)}$$

where the square of the time period originates from its one–dimensional Euclidean structure. According to the guidelines of our studies presented since the Preface, the above requirement is necessary because our sensory perceptions, even though limited, are nevertheless capable of detecting geometrically nonisomorphic realizations.

The first possible generalization meeting the above requirement is the *isotime* which we write $\hat{t} = t\times\hat{1}_t(t, ...)$, where $\hat{1}_t$ stands for the *time isounit*, i.e., the new numerical value of the unit of time, and it is assumed to be positive–definite, $\hat{1} = \hat{T}^{-1} > 0$. Despite the alteration of the unit of time, we have an isotopy inasmuch as we still satisfy the basic geometric axiom (c) of conventional time,

$$(\text{interval of time})^2\times(\text{unit}) = [\,(t_2 - t_1)\times\hat{T}\times(t_2 - t_1)\,]\times\hat{1} = (\hat{t}_2 - \hat{t}_1)^2\times 1,$$

We first learn in this way that our perception of time must not necessarily be the actual behaviour of time in biological structures, because any other time behaviour

assumption of the same time for all components of a biological system may well result to be erroneous because of the possible existence of different internal times for different components of which we essentially perceive their *average*.

In summary, the most important notion submitted in this monograph is the sequential generalizations of our perception of time into isotime, genotime and hypertime. In fact, the corresponding new methods and their implications can be derives from the above generalized times in a unique and unambiguous way.

Note that the above multidimensionality of time is grossly beyond our intuitions. Yet, it is admitted by the abstract axioms of the Euclidean geometry and, as such, it cannot be ruled out *a priory*.

We now introduce the *hypervelocity*

$$\{ \hat{v}^{>}_{k} \} = \{ \hat{d}^{>} \} \{ \hat{r}^{>}_{k} \} / \{ \hat{d}\hat{t}^{>} \} = \{ \hat{d}^{>} \} \{ \delta^{>}_{k}{}^{i} \} / \{ \hat{d}\hat{t}^{>} \} , \tag{4.12}$$

and related carrier hyperspace

$$\{ \hat{S}^{>} \} = \{ \hat{E}^{>} \} (\hat{t}^{>}, \hat{R}_{t}^{>}) \times \{ \hat{E}^{>} \} (\hat{r}^{>}, \delta^{>}, \hat{R}^{>}) \times \{ \hat{E}^{>} \} (\hat{v}^{>}, \delta^{>}, \hat{R}^{>}) , \tag{4.13}$$

where the internal brackets have been omitted for brevity, with related *total forward hyperunit*

$$\{ \hat{1}^{>}_{tot} \} = \{ \hat{1}_{t}^{>} \} \times \{ \hat{1}^{>}_{r} \} \times \{ \hat{1}^{>}_{v} \} \tag{4.14}$$

where we have assumed for simplicity $\{ \hat{1}^{>}_{r} \} \equiv \{ \hat{1}^{>}_{v} \}$.

The *forward hypernewton equations* on $\{ \hat{S}^{>} \}$, here introduced apparently for the first time jointly with the following dynamical equations of this subsection, can be written

$$\{ \hat{m}^{>} \} \frac{\{ \hat{d}^{>} \} \{ \hat{v}^{>}_{k} \}}{\{ \hat{d}\hat{t}^{>} \}} - \{ F_{k}^{>} \} (\hat{t}^{>}, \hat{r}^{>}, \hat{v}^{>}) , \quad k = x, y, z , \tag{4.15}$$

We then have the following *forward hyperlagrange equations,*

$$\frac{\{\hat{d}^{>}\}}{\{\hat{d}\hat{t}^{>}\}} \frac{\{\hat{\partial}^{>}\}\{\hat{L}^{>}\}(\hat{t}^{>},\hat{r}^{>},\hat{v}^{>})}{\{\hat{\partial}\,\hat{v}^{>k}\}} - \frac{\{\hat{\partial}^{>}\}\{\hat{L}^{>}\}(\hat{t}^{>},\hat{r}^{>},\hat{v}^{>})}{\{\hat{\partial}^{>}\,\hat{r}^{k}\}}, \tag{4.18}$$

Similarly we can introduce the hyperspace and related total forward hyperunit

$$\{\hat{S}^{>}\} = \{\hat{E}^{>}\}(\hat{t}^{>},\hat{R}_t^{>}) \times \{\hat{E}^{>}\}(\hat{r}^{>},\hat{\delta}^{>},\hat{R}^{>}) \times \{\hat{E}^{>}\}(\hat{p}^{>},\hat{\delta}^{>},\hat{R}^{>}), \tag{4.19a}$$

$$\{\hat{1}^{>}_{tot}\} = \{\hat{1}_t^{>}\} \times \{\hat{1}^{>}_r\} \times \{\hat{1}^{>}_p\} \tag{4.19b}$$

where we assume for simplicity $\{\hat{1}^{>}_p\} \equiv \{\hat{1}^{>}_r\}^{-1}$.

We also have the *forward hypercanonical hyperaction*

$$\{\hat{A}^{>}\} = \int_{t_1}^{t_2} [\,\{\hat{p}^{>}_k\}\{\hat{d}\hat{r}^{>k}\} - \{\hat{H}^{>}\}\{\hat{d}\hat{t}^{>}\}\,] =$$

$$= \int_{t_1}^{t_2} [\,\{\hat{D}^{>}_\mu\}\{\hat{d}\hat{b}^{>\mu}\} - \{\hat{H}^{>}\}\{\hat{d}\hat{t}^{>}\}\,], \tag{4.20a}$$

$$\{\hat{D}^{>}\} = (\{\hat{p}^{>}\}, \{0\}), \quad \{\hat{b}^{>}\} = (\{\hat{r}^{>}\}, \{\hat{p}^{>}\}), \tag{4.20b}$$

with ensuing *forward hyperhamilton equations* in disjoint form

$$\frac{\{\hat{d}^{>}\hat{r}^{>k}\}}{\{\hat{d}\hat{t}^{>}\}} = \frac{\{\hat{\partial}^{>}\}\{\hat{H}^{>}\}}{\{\hat{\partial}\hat{p}_k^{>}\}}, \qquad \frac{\{\hat{d}^{>}\hat{p}_k^{>}\}}{\{\hat{d}\hat{t}^{>}\}} = -\frac{\{\hat{\partial}^{>}\}\{\hat{H}^{>}\}}{\{\hat{\partial}\hat{r}^{>k}\}}, \tag{4.21}$$

or in unified notation

$$\{\omega_{\mu\nu}\}\frac{\{\hat{d}^{>}\hat{b}^{\nu}\}}{\{\hat{d}\hat{t}^{>}\}} = \frac{\{\hat{\partial}^{>}\}\{\hat{H}^{>}\}}{\{\hat{\partial}^{>}\hat{b}^{\mu>}\}}, \tag{4.22a}$$

---

selection of a reducible isounit or genounit is intrinsically contained in the formulations of Sect.s II and III, without any need of any additional clarifications, except data elaborations of specific applications.

$$i\{\hat{\partial}^{>}\}\hat{Q}/\{\hat{\partial}\hat{t}^{>}\} = \{\hat{Q},\hat{H}\} = Q\{<\}\hat{H} - \hat{H}\{>\}\hat{Q}, \tag{4.27}$$

whose brackets

$$\{A,B\} = A\{<\}B - B\{>\}A, \tag{4.28}$$

are a generalization of Albert–Santilli Lie–admissibility of Sect. 3.3 and confirm the realistic possibilities for a further generalization of the entire Lie theory, this time of multivalued character.

We finally mention that the operator hyperstructural theories are defined on the *forward hyperhilbert space* with *forward hyperstates* $\{\hat{\psi}^{>}\} = \hat{\psi}\times\{\hat{1}^{>}\}$ and *forward hyperinner product* over the forward hyperfield of hypercomplex numbers

$$\{\hat{\mathcal{C}}^{>}\}: \quad \{\hat{\psi},\hat{\Phi}\}^{>} = \{S^{-1}\}\times\int d^3r\,\{\hat{\psi}^{>\dagger}\}(r)\,\{S\}\,\{\hat{\phi}^{>}\}(r) \in \{\hat{C}^{>}\}, \tag{4.29}$$

plus the hyperstructural lifting of the remaining aspects of the isotopies and genotopies of operator formulations which we cannot possible study here.

We shall therefore content ourselves in having identified a possible multivalued generalization of the isotopic and genotopic methods, and defer further formal treatments following a verification of their significance for applications in theoretical biology.

genotopic and hyperstructural methods generally have their own intrinsic time which is different than our own perception of time, both in its rate of flow and in direction, as well as being of progressively increasing complexity depending on the complexity of the system considered.

To minimize misrepresentations, we should stress that, by no means, we claim that the above novel time behaviour is true. We only claim that it is an inevitable *consequence* of the use of the new methods which must be appraised and proved correct or erroneous in due time.

Assuming that the isotopic representation proves to be correct for certain biological structures, the additional separate problem which has to be addressed and resolve in due time is *whether we are referring to a purely mathematical property or to an actual intrinsic behaviour.*

Yet another understanding to separate science from the adaptation of science for personal needs, is that the validity for biological structure of our perception of time also cannot be claimed at this writing as being true. In fact, the resolution of the problem whether or not biological structures evolve according to a time different than our perception will predictably take generations of quantitative studies.

To illustrate the above time implications, when we observe a sea shell or a tree, it does not necessarily means that these structures really evolve according to our *perception* of time, because they may in reality have a drastically different intrinsic time. The same evidently applies also for our own body and, for this reason, we differentiate between our *perception* of time, and the intrinsic time of our own body.

The mechanism according to which biological structures may have a time behaviour different than that of our perception is that of the *isospecial relativity* of ref.s [27–29] which is based on *the alteration of the unit of time,* from our fixed and perennial unit I = +1 sec, to a well behaved, but otherwise arbitrarily positive or negative *function,* the *isounit of time*

$$I_t = +1 \text{ sec} > 0 \quad \to \quad \hat{1}_t = \text{function} > 0 \text{ or} < 0 , \qquad (5.1)$$

isotopic character of the methods, that is, their capability to preserve the geometric axioms of our perception of time, and merely provide a different *realization*. It is at this point where the selection of isotopic rather than other methods appear in its full light.

It is easy to see that the use of other methods, such as the so–called *deformations*, implies generalized structures which are no longer isomorphic to the original ones. A drastic and perhaps irreconcilable difference would then emerge between our perception and the latter mathematical representations.

As shown in Sect. 2.2, the conventional one–dimensional Euclidean space $E(t,R_t)$, representing our perception of time over the field of real numbers $R_t(t,+,\times)$, *coincides* at the abstract level with its isotopic, genotopic and hyperstructural extensions,

$$E(t, R_t) \approx \hat{E}(\hat{t}, \hat{R}_t) \approx <\hat{E}>(<\hat{t}>, <\hat{R}_t>) \approx \{ <\hat{E}>(<\hat{t}>, <\hat{R}_t>) \} . \qquad (5.4)$$

In view of the above identities, we can express at this writing only *personal opinions* on the behavior of time of biological structures one way or another, but we have *no scientific evidence* to reach definite conclusions.

We can say in an alternative way that the preservation of conventional axioms permits the interpretation that our perception of time is the *projection* of reality in our sensorial systems, and this interpretation would apply also for the multidimensional hypertime which becomes one–dimensional when projected in our sensorial systems.

The geometric treatment of the above occurrence has been illustrated in Sect. 2 via the *isobox* of Fig. 2.1 which essentially consists of the same object (the isobox which is now re–interpreted as any biological structure such as a sea shell or a tree) which is inspected by two observers, one outside observer here identified with ourselves, and internal observer identified with the intrinsic behaviour.

The main result is that *an object can exist at different present or future times and can have different times flows for the external and internal observer,* which is precisely the implication reported here.

Note that the above occurrence implies that *biological structures are*

The isotopies of the above space perception have been studied in detail in Sect. 2.2, and they imply the structural alteration of the basic unit of space into the isounits

$$I_k = +1 \quad \to \quad \hat{I}_k = \text{functions} > 0 \text{ or} < 0 , \quad k = a, y, z, \tag{5.5}$$

again under the condition

$$(\text{L} = \text{Length})^2 \times (\text{I} = \text{Unit}) = \text{invariant} . \tag{5.6}$$

Recall from Sect. 2.2 that, not only the numerical value +1 is changed, but it is changed in a different way for different axes, can assume both positive and negative values, and can vary in time.

The *isobox* of Fig. 2.2 can then again illustrate the implications in space behaviour. In fact, whatever shape we perceive from the outside, the inside shape can be different. In particular, a rather complex shape which is perceived in the Euclidean geometry can be reduced to the perfect sphere in isospace, the *isosphere* of Sect. 2.2.E). In turn, this permits the restoration of a fundamental symmetry of nature, the rotational symmetry, as being *exact* for *arbitrarily nonspherical* shapes (App. C).

Moreover, an object which is perceived as very small by us can be very large in its own isogeometry and viceversa. As an illustration, if an object is 0.03 cm in diameter *when referred to our unit of* +1 cm, it can measure 300 cm under a suitable isotopy of the unit, according to the rule

$$(0.03 \text{ cm}^2)(+1 \text{ cm}^2) \equiv (300 \text{ cm}^2)(0.0001 \text{ cm}^2) . \tag{5.7}$$

Moreover, the way we perceive the evolution in time of the shape considered can be different than the intrinsic one.

It is evident that a much greater departure from our intuition occurs for the use of genotopic methods, where we see the appearance of *four different shapes,* one per each of the four different directions of time

If our sensory perception is so transparently limited for so an elementary case as the above, the same limitations for a more realistic understanding of biological structures then become beyond *credible* doubts.

CHARACTERIZATION OF CELLS VIA ISOUNITS OF SPACE AND TIME

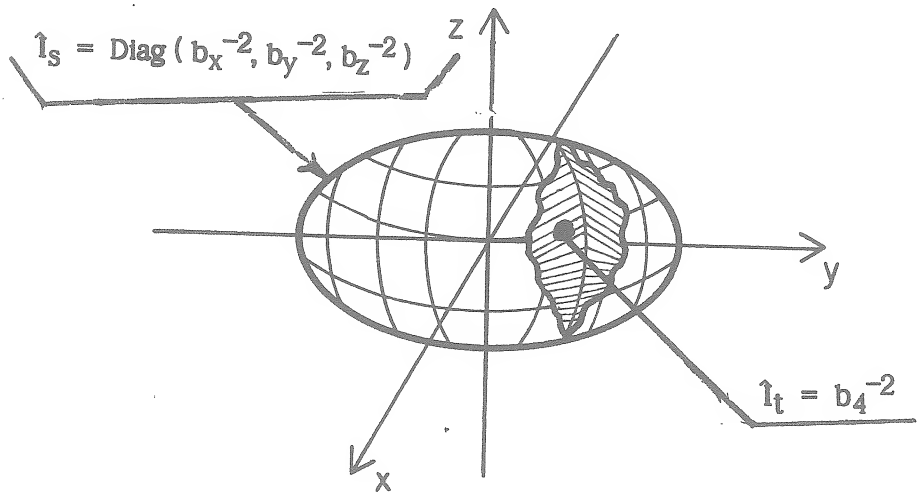

FIGURE 5.1. A schematic view of the characterization of cells via isounits of space and time. The main idea is that physical systems (e.g., those of electromagnetic type) are entirely representable via a Lagrangian or a Hamiltonian, that is, they solely admit interactions derivable from a potential. On the contrary, biological systems admits effects beyond the representational capabilities of a Lagrangian or

isoeuclidean representation [9]. In fact, in this case the isotopic element is given by the index of refraction, and the isoangle (App. A) of the stick with the water surface in the interior is equal to that outside the water. This case has been recalled to indicate again that conventional geometries apply for the exterior problem in vacuum and the isogeometries have been conceived and constructed by this author for interior problems. Moreover, their interior behaviour has been constructed in such a way to coincide with the exterior, which is the main property for the scientific credibility of the anomalies here considered.

known to this author which is compatible with our sensory perception is that via the hyperstructures with *forward hyperunits of space and time* $\{\hat{1}_a>\} = \{\hat{1}_{1a}>, \hat{1}_{2a}>, ...\}$, a = s, t, and the remaining hyperunits obtained via isoduality and isoinversions (Fig. 4.2). This yields a characterization of biological systems as being structurally more general than any physical system, equipped with the representational capabilities necessary for the description of at least some of the complexities of the biological world.

We should also not forget that simple observations of the behaviour of plants indicate quite clearly the existence of anomalous space behaviors, that is, space behaviour anomalous with respect to our *perception* of space. As one example, among so many, this author has observed a cypress growing underneath a balcony in his house which growing has now stopped for years *without touching the overhead balcony,* while cypresses are known to grow up to towering heights. Similarly, we have all observed plants from ordinary seeds grown inside bottles which stop their growth without reaching the walls, or the rather peculiar growth of plants along directions compatible with their environment. Since plants do not have eyes, the existence of a space behaviour in biological structures beyond our perception appears to be evident. The only issue which is open on scientific grounds is its correct quantitative representation.

It is finally recommendable to indicate since these preliminary lines the implications of isotopic, genotopic and hyperstructural methods for *motion in space.*

Due to use extended through centuries, we have been accustomed to the Newtonian notion that a particle initially in a point P_1 at rest with respect to us can be put in motion and moved to another point P_2 *only via the application of a force.* The implications of our studies in this respect are the following:

Implications for locomotion: *Biological structures represented with isotopic, genotopic and hyperstructural methods can perform internal motion according to the "geometric propulsion", that is, the possibility of locomotion from one point to another via the alteration of the geometry itself, without any application of a Newtonian force.*

It is evident that the above shape is indeed fully Euclidean without any need of broader geometries.

COMPUTER VISUALIZATIONS OF SEA SHELLS

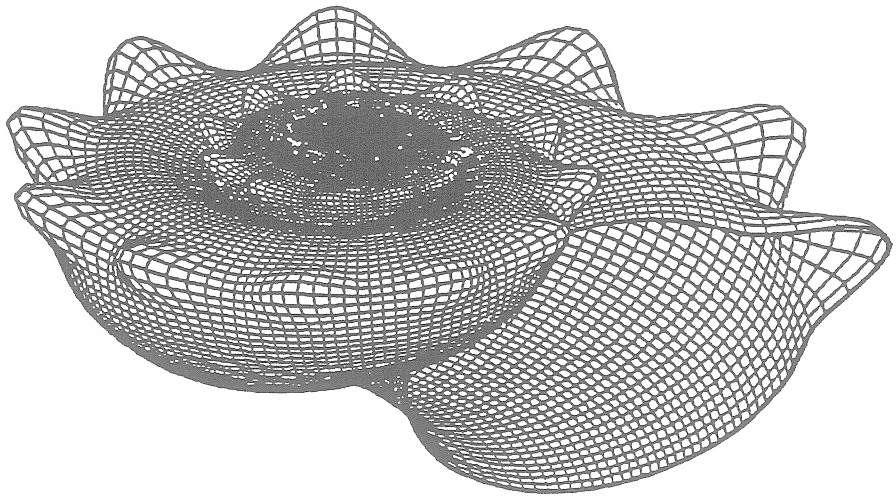

FIGURE 5.2: An example of the computer visualization of sea shells studied in Illert [45], the *Angaria Delphinius* of p. 64. Such a visualization has a rather crucial relevance for the studies reported herein. In fact, we cannot possibly "fabricate" a sea shell in laboratory according to our theories. The computer visualization is therefore the only mean available at this writing for verifying the accuracy of any given representation. The isoeuclidean representation of sea shells growth outlined below does indeed verify this condition, by showing normal growth, while the conventional Euclidean representation does not.

Generalized methods emerge to be needed for a rigorous, quantitative representation of the ***growth in time*** of sea shells. In this latter respect, Illert [loc.

and the only admissible Lagrangians must have the structure

$$L = K_1 (\psi \times \psi)^n + K_2 (\psi \times \xi)^m + K_3 \xi \times \xi)^p , \tag{5.13}$$

The property proved by Illert [loc. cit.] is that no representation of the sea shell growth consistent with evidence is possible under these conditions because the strict and rigid imposition, say, of the Euclidean product would imply in computer visualization that the sea shell first grow in a distorted way and then would crack.

Santilli [loc. cit.] has proved that the growth is instead in computer visualization under the assumption that the background geometry is of *isoeuclidean* type (Sect. 2.2). In the latter case the carrier space is the following Kronecker product

$$\hat{S}(\hat{t}, \hat{\xi}, \hat{\psi}) = \hat{E}(\hat{t},\hat{R}_t) \times \hat{E}(\hat{\xi},\hat{\delta},R) \times \hat{E}(\hat{\psi},\hat{\delta}, \hat{R}) , \tag{5.14}$$

were $\hat{t} = t\hat{1}_t$, $\hat{\xi} = \xi\hat{1}$, $\hat{\psi} = \psi\hat{1}$, with related seven-dimensional total unit

$$\hat{1}_{tot} = \hat{1}_t \times \hat{1} \times \hat{1} = \hat{1}_t \times dig. (\hat{1}_x, \hat{1}_y, \hat{1}_z) \times dig. (\hat{1}_x, \hat{1}_y, \hat{1}_z) ,$$

$$\hat{1}_k = \hat{1}_k(t, \xi, \psi, ...) = T_k^{-1}, \quad k = x, y, z , \tag{5.15}$$

isodifferentials $\hat{d}t = \hat{1}_t dt$, $\hat{d}\xi = Id\xi$, and isoderivatives $\hat{\partial}/\hat{\partial}t = T_t \partial/\partial t$, $\hat{\partial}/\hat{\partial}\xi = T\partial/\partial\xi$.

A first implication of the isoeuclidean geometry is the generalization of the scalar product into the following *isoscalar form* (Sect. 2.2.D)

$$\hat{\xi}^{\hat{2}} = \hat{\xi} \hat{\times} \hat{\xi} = (\xi_x T_x \xi_x + \xi_y T_y \xi_y + \xi_z T_z \xi_z) \hat{1} \in \hat{R} , \tag{5.16}$$

with more general expressions for nondiagonal isounits herein ignored for simplicity. The conventional distance is then lifted into the *isodistance* (Sect. 2.2.B)

$$\hat{D}_{Isoeucl.} = [(\xi_{1x} - \xi_{2x})^2 T_x + (\xi_{1y} - \xi_{2y})^2 T_y + (\xi_{1z} - \xi_{2z})^2 T_z)^{1/2}, \tag{5.217}$$

Moreover, it is possible to see the need for the representation of sea shell growth for *isotopic methods of Class III* already at this level, as illustrated in Fig. 5.2 in connection with bifurcations. Deeper studies then indicate the need in actuality of *isotopic methods of Class IV* because of the expected need to include the zeros of the isounit $\hat{1} = 0$ as a necessary condition for a smooth transition from positive to negative values of the isounit of time.

Note that the growth of sea shells is strictly *nonconservative and irreversible*. Yet the *isoeuclidean* geometry is sufficient for this simple problem, by representing irreversibility via a noninvariant time behaviour of the *isounit*, $\hat{1}[\xi(t)] \neq \hat{1}(\xi(-t)]$.

Nevertheless, the use of the *genoeuclidean geometry* (Sect. 3) appears to be technically more appropriate for a deeper axiomatization of irreversibility. This latter reformulation is left for study to the interested reader.

We should also note that the use of the genogeometries is expected to become unavoidable at bifurcations. In fact, relatively simple bifurcations can be quantitatively interpreted via *three* different time evolutions, while more general bifurcations are expected to requires *four* different motions in time, which is exactly the number of components of our genotime.

Finally, it is important to note that the results obtained by Illert [loc. cit.] indicate the need for a correct representation of *doubling each axis*, as illustrated in Fig. 5.3. This is sufficient evidence for the applicability of the *hyperstructural methods* where the isounit is a two-dimensional set.

We finally note that a mathematically more adequate representation of sea shells requires a *discrete theory* because, as studied in details in Illert [loc. cit.], the growth of sea shells is not continuous but in discrete increments. It is evident that the latter behaviour can be represented with formulations of the largest possible Kadeisvili class, those of Class V with an arbitrary generalized unit, thus including discontinuous or discrete units.

We close this section with an illustration of the alteration of shape in the transition from an exterior to an intrinsic observer. This will be done by showing that, *as a limit case*, an arbitrary shape of a sea shall can be reduced to the *isosphere* of Sect. 2.2.E, i.e., the perfect sphere in isospace. The reduction is not a

grossly outside the representational capabilities of the Euclidean geometry. The isoeuclidean geometry of Class III permits instead a quantitative representation of the bifurcations because *the isounit of time can have arbitrary values, thus being in the past or in the future, and can be either positive or negative, thus permitting motion forward or backward in the past and in the future.* It is hoped the reader can see the implications indicated in Sect. 5.1, to the effect that, when we look at a sea shell in our hands, this does not means that the sea shell is necessarily evolving with our time because it can evolve in a conceptually and structurally different time.

Consider the *hyperbolic clocksprings of the first kind in a plane,* ref. [45], Eq. (3.14), p. 81,

$$x = e^{-\frac{1}{2}\lambda\phi} \cosh[(\alpha + \tfrac{1}{2}\lambda)\Phi] \cos\phi ,$$

$$y = a\, e^{-\frac{1}{2}\lambda} \phi \sinh[(\alpha + \tfrac{1}{2}\lambda)\phi] \sin\phi . \qquad (5.21)$$

It is easy to see that in the isoeuclidean plane $E(r,\hat{\delta},\hat{R})$, $\hat{\delta} = T\delta$, $T = \text{diag.}\,(T_x, T_y)$, the above surface reduces to the perfect *isocircle.* In fact, under the values of the isotopic element

$$T_x = \{a\, e^{-\frac{1}{2}\lambda\phi} \cosh[(\alpha + \tfrac{1}{2}\lambda)\Phi]\}^{-2}, \qquad (5.22)$$

$$T_y = \{a\, e^{-\frac{1}{2}\lambda} \phi \sinh[(\alpha + \tfrac{1}{2}\lambda)\phi]\}^{-2}, \qquad (5.23)$$

the preceding equations reduce to the isopolar coordinates (App. B)

$$x = T_x^{-\frac{1}{2}} \cos(T_x^{-\frac{1}{2}} T_y^{-\frac{1}{2}} \phi) ,$$

$$y = T_y^{-\frac{1}{2}} \sin(T_x^{-\frac{1}{2}} T_y^{-\frac{1}{2}} \phi) , \qquad (5.24)$$

and they do indeed describe a perfect circle in isospace, the isocircle

$$x\, T_x\, x + y\, T_y\, y = \cos^2(T_x^{-\frac{1}{2}} T_y^{-\frac{1}{2}} \phi) + \sin^2(T_x^{-\frac{1}{2}} T_y^{-\frac{1}{2}} \phi) = 1 . \qquad (5.25)$$

$$\{ \hat{S}(\hat{t}, \hat{\xi}, \hat{\psi}) \} = \{ \hat{E}(\hat{t},\hat{R}_{\hat{t}}) \} \times \{ \hat{E}(\hat{\xi},\hat{\delta},\hat{R}) \} \times \{ \hat{E}(\hat{\psi},\hat{\delta}, \hat{R}) \},$$

and related methodology studied in Sect. 4 (hypernumbers, hyperfields, hyperdifferential calculus, etc.), where the generalized character of each isounit (rather then the trivial value +1) is expected for the representation of more complex processes. Note that the particular class of hypergeometries introduced in Sect. 4 coincide at the abstract level with the conventional Euclidean geometry. This yields a full compatibility between our *three-dimensional* perception of the sea shells and their possibly *arbitrary dimensionality in their own structures,* the above two-dimensional case being just the simplest possible one. Note also that the hyperstructures imply *a multidimensional hypertime,* that is, a time which at the abstract level coincides with our own time, yet possesses two separate components. The axiomatization of the irreversible character of the bifurcations then requires the formulation of the hyperstructures as a generalization of the genotopic methods presented in Sect. 3.

A similar result evidently occurs for the *hyperbolic clockspring of the second kind,* ref. [45], Eq. (3.15), p. 81, where we have the interchange of T_x and T_y.

Along similar lines, it is easy to see that the *Lissajous spiral,* ref. [45], Eq. (3.27), p. 90, occurring for the *nipponites,* is indeed a perfect sphere in a three-dimensional isoeuclidean space. One has to solve the following equations in r and B

$$x = a\, e^{\alpha \phi} [\, 1 + e^{\phi} \cos (2 \gamma \phi)] \cos \phi =$$

$$= r [B_{22}{}^{-1} \sin (B_{21} B_{22} \theta)] [B_{11}{}^{-1} \cos (B_{11} B_{12} \phi)],$$

$$y = a\, e^{\alpha\phi} [\, 1 + e^{\phi} \cos (2 \gamma \phi)] \sin \phi =$$

$$= r [B_{22}{}^{-1} \sin (B_{21} B_{22} \theta) [B_{12}{}^{-1} \sin (B_{11} B_{12} \phi)], \qquad (5.26)$$

$$z = b\, e^{\beta \phi} \sin (\gamma \phi) = r\, B_{21}{}^{-1} \cos (B_{21} B_{22} \theta),$$

in which case the shells is represented via the isospherical coordinates (App. B) as the perfect sphere in isospace

5.3. Apparent isotopic origin of valence

In the preceding subsection we have illustrated the possibilities in theoretical biology of *classical* generalized methods. It is important now to illustrate the possibilities for their *operator* counterpart.

Among various possible applications of the latter type that we believe to be truly fundamental for theoretical biology is the study of the apparent isotopic structure of the bonding of atoms into molecules, here generically referred to the valence. In fact, the bonding of atoms is a true, ultimate structure of biology at large, whether molecular or of other character.

It should be stressed that the scientific value of the currently available *quantum mechanical* representation of the valence (see, e.g., [1,2,3]) is unquestionable. Nevertheless, quantitative sciences essentially provide successive *approximations* of physical, chemical or biological structures. Thus, there are reasons to expect that the current representation of the valence will not resist the test of time as the final theory and be only the first of deeper representations.

This is due to the following facts. The bonding of atoms into molecules is essentially realized by the peripheral *electrons*. But electrons *repel* each other according to quantum mechanics. The reader can therefore see the emergence in the valence of a fundamental problem: how can electrons, which normally repel each other, allow the bonding of atoms which requires attraction ?

As we shall see in this subsection, the isotopies of quantum mechanics (QM), also known as *hadronic mechanics* (HM) outlined in Sect. 2.3.F and studied in details in monographs [9,10] do indeed permit a quantitative solution of the above problem. Such a solution was first identified by Santilli in the original proposal to build hadronic mechanics [6b] , where he showed that an electron e^- and a positron e^+ can form a new bound state in singlet coupling at short distances (of the order of 1 fm = 10^{-13}cm) identified with the π° meson and represented with the script $\pi^\circ = (e^-_\uparrow, e^+_\downarrow)_{HM}$. This result was submitted as the first application of hadronic mechanics. In fact, the sole bound state of an electron and a positron permitted by quantum mechanics is the positronium $= (e^-_\uparrow, e^+_\downarrow)_{QM}$.

The basic mechanism submitted in memoir [6b] is the *nonlocal-integral and*

known. We are interested in the physical reality in which there is *attraction* among *identical* valence electrons. The turning of attraction into repulsion evidently requires a new wavefunction hereon denoted $\hat{\psi}_\uparrow(t, r)$ and assumed to be an isostate of an isohilbert space $\hat{\mathcal{H}}$.

By recalling that quantum mechanical Coulomb interactions are invariant under unitary transforms, the only possibility for the map $\psi_\uparrow \to \hat{\psi}_\uparrow$ is to be represented by a *nonunitary* transform $\hat{\psi} = U\psi$, $UU^\dagger = U^\dagger U = \hat{1} \neq I$, where $\hat{1}$ has yet to be determined (see below). This activates *ab initio* the applicability of hadronic mechanics because, by conception [6b] and realization [10] the latter is a nonunitary image of the former. The first step of the proposed model is therefore that of transforming system (5.30) in $\psi_\uparrow$ into a new system in $\hat{\psi}_\uparrow = U\psi_\uparrow$ where U is nonunitary,

$$U\, H_{Coul} U^\dagger (U\, U^\dagger)^{-1}\, U_{\cdot}\, \psi_\uparrow(t, r) = \hat{H}_{Coul.}\, T\, \hat{\psi}_\uparrow(t, r) =$$

$$= \{\frac{1}{2m}\, \hat{p}_k\, T\, \hat{p}^k + \frac{e^2}{r}\, \hat{1}\}\, T\, \hat{\psi}_\uparrow(t, r) = E\, \hat{\psi}_\uparrow(t, r),$$

$$\hat{p}_k\, T\, \hat{\psi}_\uparrow(t, r) = -\, i\, \hat{1}_k{}^i\, \partial_i\, \hat{\psi}_\uparrow(t, r), \qquad (5.31)$$

However, the above system is incomplete for the description of the valence, because the valence electrons are not isolated in space, and are instead immersed in the positive field of the nuclei hereon denoted N^+ and represented by the familiar term $-ze^2/r$. Being external, the latter term is not transformed (i.e., it remains conventionally quantum mechanical) and therefore it should be merely added to the transformed equations (5.31).

The formal equations of the *isotopic model of valence* here proposed are therefore given by

$$\{\frac{1}{2m}\, \hat{p}_k\, T\, \hat{p}^k + \frac{e^2}{r}\, \hat{1} - z\, \frac{e^2}{r}\}\, T\, \hat{\psi}_\uparrow(t, r) =$$

$$= \frac{1}{2m}\, \tilde{p}_k\, T\, \tilde{p}^k\, T\, \hat{\psi}_\uparrow + \frac{e^2}{r}\, \hat{\psi}_\uparrow - z\, \frac{e^2}{r}\, T\, \hat{\psi}_\uparrow(t, r) = E\, \hat{\psi}_\uparrow(t, r),$$

isoschrödinger equation can be written

$$\{ -\frac{\hat{1}}{2\hat{m}} r^2 \frac{d}{dr} r^2 \frac{d}{dr} - (z-1)\frac{e^2}{r} - V_o \frac{e^{-r/R}}{1-e^{-r/R}} \} \hat{\psi}_{\uparrow}(r) = E\, \hat{\psi}_{\uparrow}(r) \tag{5.38}$$

where $\hat{m}$ is the *isotopic mass* (i.e., the image of m in isospace essentially given by m multiplied by a scalar which provides a sort of renormalization).

The solution of the above equation is known from Santilli [6b], Sect. 5.1. The Hulten potential behaves at small distances like the Coulomb potential,

$$V_{Hulten} = V_o e^{-r/R}/(1 - e^{-r/R}) \approx V_o R/r. \tag{5.39}$$

At distances smaller than the coherent length of the valence pair, Eq. (5.38) can therefore be effectively reduced to the form

$$\{ -\frac{1}{2\hat{m}} r^2 \frac{d}{dr} r^2 \frac{d}{dr} - V \frac{e^{-r/R}}{1-e^{-r/R}} \} \hat{\psi}_{\uparrow}(r) = E\, \hat{\psi}_{\uparrow}(r), \tag{5.40}$$

where $V = V_o R + (z-1)e^2$ with general solution, boundary condition and related spectrum (ref. [6b], pp. 837–838)

$$\hat{\psi}_{\uparrow}(r) = {}_2F_1(2\alpha + 1 + n, 1-\alpha, 2\alpha + 1, e^{-r/R})\, e^{-\alpha r/R} (1 - e^{-r/R})/r,$$

$$\alpha = (\beta^2 - n^2)/2n > 0, \qquad \beta^2 = \hat{m} V R^2/\hbar^2 > n^2,$$

$$E = -\frac{\hbar^2}{4\hat{m}R^2} \left(\frac{\hat{m} V R^2}{\hbar^2} \frac{1}{n} - n \right)^2, \quad n = 1, 2, 3, \ldots \tag{5.41}$$

where we have reinstated $\hbar$ for clarity.

Santilli [6a] identified the numerical solution of Eq.s (5.38) for the hadronic model $\pi^\circ = (e^+_{\uparrow}, e^-_{\downarrow})_{HM}$ (in which there is evidently no contribution from the nuclear field to the constant V), by introducing the parameters

$$k_1 = \hbar/2\hat{m}Rc_o, \qquad k_2 = \hat{m}VR^2/\hbar, \tag{5.42}$$

solely of hydrogen, have only one baryon, the proton, and no representation is possible via quantum mechanical quark models because they would require the use of at least seven additional baryons which do not yet exist) [9,10]. It should be stressed that *none* these models is possible within the context of quantum mechanics.

A most interesting feature of the above *isotopic structure model of hadrons with ordinary massive particles as physical constituents* is that *it suppresses the spectrum of the Hulten potential down to only one admissible level,* the hadron considered. This feature was identified since the original proposal [6b] and it is called the *hadronic suppression of atomic spectra.*

For clarity, it should be indicated that, in reality, the spectrum does indeed remain infinite. For instance, in the e^+–e^- model the spectrum is composed by the conventional atomic spectrum of the positronium described by quantum mechanics (which is recovered identically for non–appreciable overlappings of the wavefunctions), plus *one single additional* bound state structural beyond any realistic capability of quantum mechanics, which is described by the covering hadronic mechanics, and represents the π°. Therefore, the remaining mesons *are not* given by excited states of the isotopic π° model, but by different constituents, e.g., $\pi^\pm = (e^+, e^\pm, e^-)_{HM}$, in which the constituents are again produced free in the spontaneous decays with the lowest rate.

The same occurs for the hadronic model $n = (p^+,e^-)_{HM}$ which is manifestly impossible for quantum mechanics, yet it is capable of representing in the covering hadronic mechanics all characteristics of the neutron, such as total rest energy, mean life, charge radius, space and charge parity, magnetic moments, etc., *as well as* the decay mode $n \to p^+ + e^- + \bar{\nu}$.

The reader should be aware that the novel isotopic structure of the neutron $n = (p^+,e^-)_{HM}$ as synthesized in stars is expected to have significant implications in theoretical biology, as we hope to indicate in some future, more specialized papers. After all, at the quantum mechanical level the neutron preserves its identity, but a deeper level its existence is eliminated in favor of protons and electrons under nonlinear, nonlocal and nonhamiltonian interactions.

self-evident implications in theoretical biology, as we also hope to indicate in some future paper. After all, the model goes at the very structure of the molecular chains.

ISOTOPIC ORIGIN OF THE COOPER PAIR

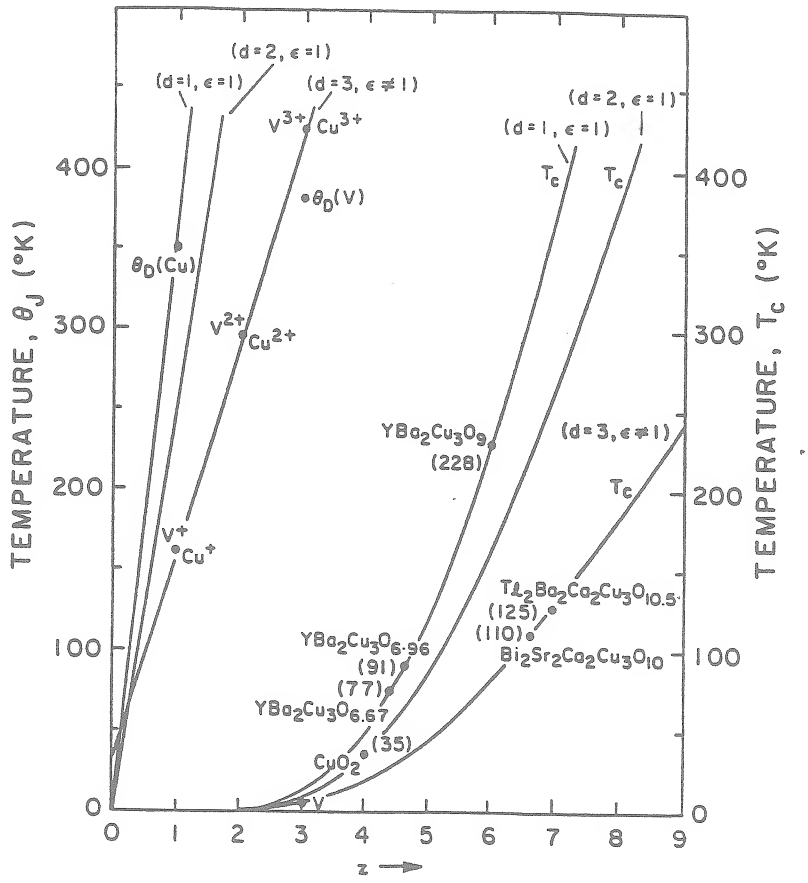

FIGURE 5.5: A reproduction of Fig. 10 of ref. [43] illustrating the remarkable agreement between the predicted dependence of T_C from the effective valence z of ions (continuous curve) and the experimental values on the "jellium temperature" for various compounds

Isotopic model of atomic bonding. It is easy to see that the use of the above results yields the following general form of the k–parameters for the case of the bonding of two atoms

$$k_1 = A\, z^{-1/2} \times 10^{-n}, \qquad k_2 = B\, z^{1/2}\ , \tag{5.51}$$

where A, n and B are numerical constants specific for each atom, and z can be assumed in first approximation as the average charge of the two nuclei.

It is also easy to see that: 1) $k_1 \ll 1$, $k_2 > 1$, 2) the model can provide the fit of known binding energies [1,2,3], including the fit for very low energies (because the ultimate mechanism is that of contact interactions which have no potential energy, the energies entering in the model being mostly kinetic); and 3) the model also produces *one single and unique* energy level for each bonding.

The application of the above results to specific cases of the bonding of atoms in molecules is straightforward, and will be done in a more specialized paper.

We therefore conclude by saying that *hadronic mechanics can indeed provide a quantitative representation of the bonding of atoms in molecules in which the identical valence electrons experience attraction.* The ultimate origin of the representation is given by the *nonlinear, nonlocal and nonhamiltonian interactions among the valence electrons*, that is, interactions which are conceptually, technically and practically beyond any realistic possibility of quantum mechanics.

The main mechanism is again the dominance over the repulsive Coulomb force of the attractive nonlinear, nonlocal and nonhamiltonian interactions at the effective valence distance. An important role is played in the model by the nuclei which are responsible for the *nonlinear* character of the model (i.e., nonlinearity in the wavefunction). The *nonlocal* character originates from the mutual wave–overlapping of the electrons at short distances. Finally, the *nonhamiltonian* character is due to the fact that the overlapping of the wavepackets is a *contact zero–range* interaction for which the Hamiltonian has no conceptual, mathematical or physical meaning. Note that, whenever the wave–overlapping is no longer appreciable, i.e., for $< \hat{\psi}_\uparrow \mid \hat{\psi}_\downarrow > = 0$, $\hat{1} \equiv I$, quantum mechanics is recovered

multielectrons isostates $\hat{\Psi}(r_1, r_2, ...)$ are then constructed via the usual rules; the isounits $\hat{1}$ and isotopic elements T are now the tensorial products of individual quantities; the *electron isodensity* is then given by

$$\hat{\rho}(r) = \int \hat{\Psi}^{\dagger} T \hat{\Psi} \, d\mu \, , \tag{5.54}$$

where $d\mu$ is the applicable measure, and it is restricted by the condition

$$\int \hat{\rho}(r) \, dr = N \, , \tag{5.55}$$

where N is the total number of electrons in the considered volume.

The reader should be aware that the *totality* of quantum mechanical methods must be isotopically lifted for consistency without any exception known to this author (otherwise there are major inconsistencies, such as lost of invariance, lack of Hermiticity, etc.). This implies also the lifting of the perturbation theory (see Ch. 11 of ref. [10] for brevity). The energy change for one-electron in first-order isoperturbation is then given by

$$E_k^{(1)} = \int \hat{\Psi}_k^{0} T \Delta V T \hat{\Psi}_k^{0} \, d\mu \, , \tag{5.56}$$

with isoperturbed wave function

$$\hat{\Psi}_k^{1} = \hat{\Psi}_k^{0} + \sum_{j \neq k} < \hat{\Psi}_j^{0} | T \Delta V T | \hat{\Psi}_k^{0} > / (E_k^{0} - E_j^{0}) \, , \tag{5.57}$$

and related change in the isodensity functional. The isotopic lifting of the remaining aspects is then consequential. The isotopies of the remaining aspects , such as the Kohn-Sham model is then consequential.

The most significant advance in the above isotopic model over the conventional one is that the isostates $\hat{\psi}$ and $\hat{\Psi}$ are indeed isostates of an *attractive* interaction among the atoms as outlined in this section. Therefore, the isotopies complement conventional results with such a missing property, and imply the

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deformable, while physical systems are (generally assumed to be) *rigid*. Similarly, biological systems are composed of constituents, such as cells, in mutual physical *contact* among each others, while physical systems such as an atomic system, are made up of particles with action-at-a-distance interactions. It then follows that biological systems are expected to have novel, internal *zero-range interactions* which are unthinkable in contemporary theoretical physics. Moreover, contact interactions among cells are *nonlocal-integral* because extended over a finite surface, while physical systems are notoriously assumed to be of local-differential type, that is, reducible to a finite set of isolated points.

All these aspects evidently requires the identification of new methods, more general than those of contemporary physics which are more suitable for quantitative studies of biological systems.

The basic role of irrebersibility. The aspect which ultimately rendered compelling the writing of this monograph is the problem of *irreversibility*. In fact, physical systems such as planetary or atomic systems, are *reversible*, i.e., their image under the reversal of the direction of time is as admissible as the original system. On the contrary, all known biological systems are *irreversible*, i.e., their image under time reversal cannot occur in reality (e.g., a human would go back to its embryo under time reversal).

On the other side, the discipline of general use in theoretical biology, quantum mechanics, is notoriously *reversible*. This creates rather serious problems of internal consistency in current studies of theoretical biology, particularly in view of the following

NO REDUCTION THEOREM [b]: *Under sufficient topological conditions, classical, macroscopic, irreversible systems cannot be consistently reduced to a finite collection of particles all in reversible conditions and, vice-versa, a finite collection of particles all in reversible conditions cannot consistently yield a macroscopic irreversible system.*

Stated in a nutshell, the above theorem establishes that *quantum mechanics*

In the latter respect the reader should be made aware of the many inconsistencies of numerous generalizations of quantum mechanics attempted during the second half of this century, such as the so–called q–, k– and quantum deformations, nonlinear theories (i.e., nonlinearity in the wavefunctions), and others. To see this occurrence, recall that, to be really novel, a formulations must be outside the equivalence class of quantum formulations, that is, it must be derivable from the latter via *nonunitary transforms.*

A litany of inconsistencies then follows because all the beautiful axiomatic properties of quantum mechanics crumble under nonunitary transforms. To begin, nonunitary transforms do not leave invariant the basic unit, thus voiding the theory of any possible application to actual measurements; the same nonunitary transforms do not preserve Hermiticity in time, thus voiding the theory of any observable; no numerical data can possibly be invariant under nonunitary transforms, thus voiding the theory of any any possible numerical prediction; etc.

Particularly insidious is the use in theoretical biology of nonliner theories. In fact, they violate the superposition principle as well as other axioms of quantum mechanics (such as causality). As a result, their predictions for composite systems such as a biological structure have ne known scientific value.

Perhaps even more insidisous is the use of other broadening of quantum mechanics, such as those based on the so–called "imaginary potential" or an external term. In fact, the latter theories imply the loss, not only of all possible Lie algebras, but actually of all possible algebras in the brackets of the time evolution (technically we have the violation of the right or left scalar and distributive laws). Under these conditions, statements such as "protons and neutrons with spin 1/2" have no mathematical or physical meaning because of the loss of all possibilities do define spin.

In conclusions, rather voluminous studies conducted in recent decades have established that *any departure from the linear, local and potential characters of quantum mechanics leads to numerous inconsistencies whenever treated with conventional mathematical methods, e.g., conventional fields, conventional vector and Hilbert spaces, conventional algebras, etc.*

axiomatic properties of quantum mechanics; 2) are outside its class of equivalent; and 3) admit the latter as a particular case when the mutual distances are such to render ignorable all non-quantum-mechanical effects.

After considerable search, the general methods verifying the latter requirements resulted to be the so-called *isotopies* which are maps (also called *liftings*) of any given linear, local, and unitary structure into their most general possible nonlinear, nonlocal and nonunitary forms, yet capable of reconstructing linearity, locality and unitarity in certain generalized spaces defined over suitably generalized numbers.

Because of the latter property, the isotopies are *axiom-preserving* by conception and realization, that is, whatever property exists for the original structure, it also exists for the broader structure. Unlike conventional studies, the isotopies therefore permit the preservation of the invariance of the unit, the Hermiticity of all observable, the invariance of numerical predictions, and all other characteristics of quantum mechanics, only treated in a more general space over more general numbers.

The ultimate roots of the generalized methods of this monographs therefore rest on a *generalization of the notion of numbers* and, at a deeper level, emerge from a *generalization of the unit* which has remained the value +1 since biblical times. The broadening of the remaining formulation is merely consequential.

In my original proposal of 1978 [c] the isotopies were proposed as a particular case of the so-called *genotopies* which are still more general maps such to violate the original axioms in favor of covering properties and, in this sense, they were said to be *axiom-inducing.*

With the passing of time I discovered that the axiom-inducing character of the genotopies holds only when projected in *conventional* spaces over conventional fields, while the genotopies are also axiom-preserving when defined over appropriate fields and numbers. In this way, the genotopies became a more general formulation of the isotopies. The main novelty with respect to isotopies is that genotopies require an *ordering of the multiplication, one to the right and one to the left, with consequential ordering of the unit.* The genotopies then represent rather naturally Eddington's "time arrows" and permits a characterization of

easy to see that *all these generalizations have to be excluded on grounds of our sensory perception,* evidently because our senses are able to perceive a geometry not isomorphic to the Euclidean one.

It is at this point that I realized the isotopic, genotopic and hyperstructural methods as being uniquely suitable for biological structures precisely in view of their axiom–preserving character. In fact, in inspecting a sea shell in our hands, our senses are completely unable to distinguish whether the internal geometry is Euclidean or iso–, geno– or hyper–Euclidean, precisely because the geometric axioms are the same and only realized in different ways.

In different terms, the restriction of all possible generalized methods to be of axiom–preserving character was original done on pure grounds of mathematical and phenomenological consistency, but later on emerges as possessing much more powerful values via–a–vis the compatibility with our sensory perception.

Once the latter compatibility is established, there is the emergence of an entirely new scientific horizon of possibilities which may appear to some beyond the most advanced science fiction, such as causal motion backward in time as necessary in bifurcations, a new form of locomotion within biological systems which, being based on nonpotential interactions, occurs without any Newtonian force or use of energy, and others, which are presented in this monograph for the sole attention of biologists with a young mind of all ages.

The baffling properties of biological systems. As a physicist, I cannot close this preface with an aspect which still baffles me considerably. Irreversibility is not a sole feature of biological systems, because it is also a general characteristics of physical systems, e.g., those with entropy. In fact, the "No–Reduction Theorem" quoted earlier was originally formulated [b] for *physical* systems. It implies that the origin of irreversibility, e.g., in interior gravitational problems such as quasars, must be seen at the ultimate particle level [e]. as a result, the isotopic, genotopic and hyperstructural methods have emerged as being significant also for physics [c].

After a long scientific journey, I stumbled again into my original problem of differentiating biological and physical systems. In fact, rather than being solved by my studies, the problem is more open than ever, because, even though generalized,

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Particular thanks are due to: H. Tributsch for inspiring the writing of this monograph with his seminal lecture on irreversibility; D. S. Sourlas, Gr. T. Tsagas and T. Vougiouklis for invaluable mathematical assistance; and to C. A. C. Dreismann, W. D. Lakin and L. Pohlmann for a critical reading of preceding drafts of the manuscripts.

Needless to say, I am solely responsible for the contents of this monograph due to the numerous changes in its final version.

representation of *the general instability of individual orbits, the time-rate-of-growth (or of decay) of given quantities,* and the like.

The second important limitation of contemporary methods for theoretical biology is that they are reversible in time, that is, their analytic, geometric and algebraic structures are time reversal invariant. Their use is therefore prohibited by the *No-Reduction Theorem* reviewed in the Preface, which prevents the consistent reduction of irreversible systems to a collection of reversible ones.

On the contrary, *biological systems are structurally irreversible,* that is, they require mathematical methods which are irreversible irrespective of whether the Lagrangian or Hamiltonian is time-reversal invariant. This is evidently due to the fact that the restriction of irreversibility to a long-range potential simply has no sufficient connection with a reality which is "structurally irreversible", that is, irreversible irrespective of the behaviour in time of any potential.

The remaining and perhaps deeper insufficiencies of contemporary mathematical methods for theoretical biology are of technical nature. The ultimate mathematical essence of contemporary Lie's theory is its topology which is well known to be *local-differential,* that is, it applies in the neighborhood of *a finite set of isolated points.* This topological structure re-emerges in contemporary analytic mechanics, which, as it is well known, it is solely applicable to the characterization of *a finite set of isolated massive points.* As an example, the representation of Jupiter in its trajectory in the Solar system requires its necessary representation as a massive point, as originally conceived by Galileo Galilei.

The same topological limitations re-emerge in their entirety in quantum formulations which are notoriously applicable solely for the characterization of particles, again, as *massive or massless points.* For instance, it is well known that protons and neutrons constituting a given nucleus are represented as points in first quantization.

Under certain limits of applicability, the above point-like approximation of extended particles is indeed fully effective for physical systems. As an example, the actual size, density and mass of Jupiter have no impact in its trajectory in the

primary purpose of representing *conservative systems* such as a planetary or an atomic system and they are reducible to a primitive symmetry, the celebrated G*alilei symmetry* for the nonrelativistic treatment and the *Poincaré symmetry* for the relativistic case. A necessary condition for the applicability of these symmetries and related methods is the presence of the *Keplerian nucleus,* that is, the heaviest particle occupying the center. These systems are also *stable* in time, thus being characterized by *conservation laws* of total physical quantities (energy, angular momentum, etc.). Finally, the systems are *reversible,* that is, their image under time–reversal is as possible as the original system. By comparison, biological systems, here schematically represented as a collection of cells, *do not admit any Keplerian nucleus* because an arbitrary cell can be at the center, if the center itself exists at all. The fundamental symmetries of contemporary physics are not therefore applicable to theoretical biology on this ground alone. Moreover, biological systems are *not stable,* in the sense that they either grow or decay, thus requiring their description via *time–rates–of–variations* of their own characteristics (size, shapes, weight, etc.). Finally, biological systems are *irreversible,* for the evident reason that their under time–reversal does not occur in our reality. These occurrences establish rather forcefully the need of new methods specifically conceived for theoretical biology.

By comparison, the representation of biological systems as a finete set of isolated points is manifestly insufficient because it would imply that, e.g., the ganglia of a neural network are dimensionless points, or that a cell is constituted by isolated points, each cell being connected to the rest of the structure via an external potential.

Even greater technical insufficiencies emerge in a deeper examination of the applicability of current mathematical methods to theoretical biology. For instance, contemporary geometries such as the Euclidean or Minkowskian geometry, can effectively represent only *perfectly spherical and perfectly rigid shapes.* This is due to their local rotational symmetry SO(3) which is well known to be a theory of a rigid body solely leaving invariant the perfect sphere. At any rate, the very axioms of the Minkowskian geometry have been known to be incompatible with the deformation theory throughout this entire century.

Again, the above features of contemporary methods are indeed effective for the arena of their conception. For instance, the conception of a proton or a neutron as perfectly spherical and perfectly rigid particles does indeed constitute

But, above all, the technical insufficiency of contemporary mathematical methods for theoretical biology which this author considers the most fundamental is their *local-differential character*, while the reality of biological structure is their manifest *nonlocal-integral character*, that is, the need for representing systems and their interconnections via actions in a finite volume which, as such, is not effectively reducible to a finite number of isolated points.

THE ROLE OF NONLOCAL INTERACTIONS

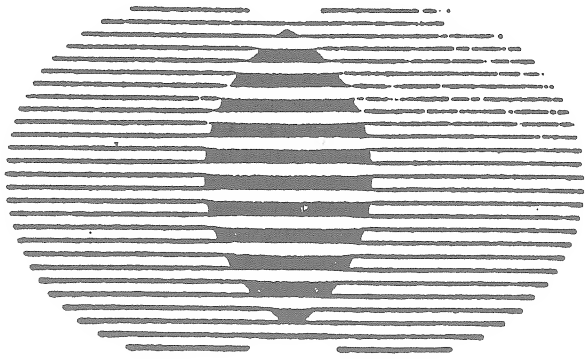


FIGURE 1.2. Contemporary physical systems are generally assumed to be *local-differential,* that is reducible to a finite set of isolated points. This assumption is necessary for the reduction of their interactions to those admitting of a potential energy. Biological systems require a structurally more general notion of interactions, hereon called *nonlocal-integral* because they are generally due to the mutual penetration and overlapping of the wavepackets of particles, atoms, and molecules. As such, the latter interactions cannot be exactly reduced to a finite set

differential type). As such, the latter representations (which are often called "indirect") will be ignored throughout our analysis. No representation of biological will be considered in this monograph unless it exists first in the coordinate system of the observer ("direct representation").

cases. Quantitative sciences do not generally advance in disjoint steps, but instead via a sequence of progressive advances, each one including the preceding one as particular case. This historical rule also has clear *experimental* values. In fact, it restricts the advances to new effects over pre–existing knowledge which must be subjected to experimental verification. *Rather than abandoning contemporary mathematical methods altogether, we shall assume them as our foundations* (see the Preface), and seek their generalization into a form suitable for the identification of new effects outside the predictive capacities of contemporary Lagrangian or Hamiltonian, classical and quantum mechanics.

2) The new methods must be structurally nonconservative. This second requirement essentially restricts the possible generalizations to those capable of representing *time–rate–of–growth (or of decay) of a given biological quantity.* In particular, the condition requires a suitable structural generalization of the conventional time evolution laws, whether classical or quantum mechanical. Since these laws are notoriously characterized by a one–parameter Lie group, the condition requires an inevitable structural generalization of Lie's theory.

3) The new methods must be structurally irreversible. This condition has implications much deeper than the preceding ones, inasmuch as it requires first a *nonlagrangian or nonhamiltonian generalization* of contemporary methods, and then *a new conception of time itself,* that is, the possible differentiation between our perception of time and the actual time of a biological structure, as we shall see.

4) The new methods must represent extended, nonspherical and deformable biological structures with nonlocal-integral interconnections. The latter conditions imply further restrictions, this time of methodological character beginning with new notions of *symmetries of extended, nonspherical and deformable structures,* and then passing to *new nonlocal–integral topologies, geometries, algebras and mechanics.*

5) The new methods must preserve the basic abstract axioms of conventional methods. This is perhaps the most unexpected condition for the noninitiated reader. Yet the condition is as important as all preceding ones for technical reasons we can study in detail during the course of our analysis and

a consequence of the reduction of all possible interactions to those derivable from a potential. On the contrary, biological systems are expected to admit internal locomotion *without any application of Newtonian forces*. This is expected to be a necessary consequence of the existence of nonpotential interactions within biological systems (Fig. 1.2). As a first rudimentary illustration, one may think of the motion of a balloon in atmosphere which does indeed move due to contact interactions also without any potential energy. Another objective of this monograph is a study of the new locomotion as due to nonlocal and nonpotential interactions which, as such, is considerably more complex than the motion of a balloon in atmosphere. As we shall see, all the differences between physical and biological systems indicated in these introductory lines appear to be due to *different units of space and time*. In this monograph we shall therefore submit the hypothesis that physical and biological systems are primarily differentiated by their *geometries* which, in turn, are differentiated via their units of space and time, rather than conventional means, such as increase of dimensions, addition of curvature, etc.

At this introductory stage we can indicate that the condition is needed from the expectation that the complexities of the biological world are dramatically beyond the simplistic possibilities of human perception. For instance, when we inspect a sea shell in our hand we perceive it on the basis of our three Eustachian tubes, mathematically represented via the Euclidean geometry. However, as we shall see, such a geometry is structurally unable to permit a quantitative representation of the *growth* of sea shells, thus requiring its structural generalization. At this point, the studies conducted by this author have indicated that drastic geometric changes from our perception generally lead to inconsistencies. The best solution of this problem known to this author is the preservation of the basic axioms of the Euclidean geometry at the abstract level, thus essentially reducing the new formulations to more general *realizations* of the same abstract axioms.

This monograph is devoted to an outline of the novel mathematical methods verifying the above requirements 1)–5) with particular emphasis on their specialization to biological structures. The presentation will be intentionally restricted to a form as elementary as possible, so as to be understood by a broader audience. Readers primarily interested in mathematical rigor are suggested to conduct mathematical studies in mathematical Journals.

generally nonlinear and nonlocal dependence on all needed quantities and their derivatives, such as the coordinates $r = (x, y, z, ...)$ and wavefunctions $\psi(t, r)$, their derivatives of arbitrary order, $\dot{r}, \ddot{r}, \partial\psi, \partial\partial\psi$, as well as the local temperature τ, the local density μ, the local index of refraction n, etc., and we shall write the map

$$I = \text{diag.}\,(1, 1, 1, ...) \quad \to \quad \hat{1} = \hat{1}(t, r, \dot{r}, \ddot{r}, \psi, \partial\psi, \partial\partial\psi, \tau, \mu, n, ...) = \hat{1}^{\dagger}, \qquad (1.1.2)$$

which is called *isotopic lifting* or , just *lifting* for short.

As we shall see, the preceding lifting can represent most of aspects 1)–4), because it introduces the new quantity $\hat{1}$ *in addition to* the conventional Lagrangian or Hamiltonian. As such, $\hat{1}$ can represent extended, nonspherical and deformable shapes, as we shall see. Similarly, the conventional Lagrangian or Hamiltonian can represent conservative terms, while the new quantity $\hat{1}$ can represent nonconservative effects. The representation of irreversibility is evident for $\hat{1}(t, ...) \neq \hat{1}(-t, ...)$ even when the Lagrangian and Hamiltonian are time–reversal invariant. Finally, the representation of nonlocal effects is also transparent from integral realizations of $\hat{1}$, as we shall see in the applications.

However, lifting (1.1.2) alone violates condition 5) on the preservation of the original axioms. In order to verify the latter condition, Santilli [6] introduced the additional joint lifting of the conventional associative product $A \times B = AB$ among generic quantities A, B into the new product

$$A \times B = A\,B \quad \to \quad A \hat{\times} B = A \times \hat{T} \times B\,. \qquad (1.1.3)$$

Under the condition that

$$\hat{1} = \hat{T}^{-1}, \qquad (1.1.4)$$

$\hat{1}$ is indeed the correct right and left unit of the new theory,

$$\hat{1} \hat{\times} A = \hat{T}^{-1} \times \hat{T} \times A \equiv A = A \hat{\times} \hat{1} = A \times \hat{T} \times \hat{T}^{-1}, \qquad (1.1.5)$$

The fundamental dynamical equations of the isotopic theory are then given by the expression in infinitesimal form

$$i\,dA / dt = [A, H] = A \hat{\times} H - H \hat{\times} A = \tag{1.1.8}$$

$$= A \times \hat{T}(t, r, \dot{r}, \ddot{r}, \psi, \partial\psi, \partial\partial\psi, \tau, \mu, n, ...) \times H - H \times \hat{T}(t, r, \dot{r}, \ddot{r}, \psi, \partial\psi, \partial\partial\psi, \tau, \mu, n, ...) \times A$$

with exponentiated form

$$A(t) = (e^{iH\times\hat{T}\times t}) \times A(0) \times (e^{-it\times\hat{T}\times H}), \tag{1.1.9}$$

first introduced in memoir [6b], which constitute a *Lie–Santilli isogroup* [11–14].

The above equations outline the structural elements of the isotopies of Lie's theory, first introduced in memoir [6a], today called *Lie–Santilli isotheory*, and referred to the isotopies of enveloping algebras, Lie algebras, Lie groups, representation theory, etc.

The name "isotopies" was selected by Mrs. Carla Santilli and quoted in memoirs [6a] from its Greek meaning of "preserving configurations" and interpreted as "axiom preserving". Today, *the isotopies refer to maps of any given linear, local and Hamiltonian structure into its most general possible nonlinear, nonlocal and noncanonical extension which are, however, such to reconstruct linearity, locality and canonicity on certain generalized spaces over generalized fields.*

In fact, dynamical equations (1.1.8) and (1.1.9) are highly nonlinear, nonlocal and nonhamiltonian, yet they are *isolinear, isolocal and isocanonical,* that is, verifying the conditions of linearity, locality and canonicity in isotopic spaces, as we shall see.

In short, the isotopies are based on a generalization of the basic unit of the conventional methods. This requires for consistency the lifting of the *totality* of conventional mathematical methods, including numbers, fields, angles,

under the interconnecting properties

$$R = S^{\dagger}, \tag{1.1.14}$$

and conditions

$$\hat{1}^{>} = S^{-1}: \quad \hat{1}^{>} > A \equiv A = A > \hat{1}^{>},$$

$$\hat{1}^{<} = R^{-1}: \quad A < \hat{1}^{<} \equiv A = \hat{1}^{<} < A. \tag{1.1.15}$$

When (and only when) all the above conditions are met, $\hat{1}^{>}$ ($\hat{1}^{<}$) is called the *forward genounit (backward genounit),* and $\hat{R}$ ($\hat{S}$) is called the *forward genotopic element (backward genotopic element).*

In the conventional formulation of Lie algebras, the antisymmetric algebra ξ^{-} with product $A\times B - B\times A$ is made up of two contributions, the first for the multiplication to the right $\xi^{\rightarrow}$: $A\rightarrow B = A\times B$ and the second for the multiplication to the left $\xi^{\leftarrow}$: $B\leftarrow A = B\times A$ with Lie product

$$[A, B] = A \rightarrow B - B \leftarrow A \equiv A \times B - B \times A, \tag{1.1.16}$$

in which case we have the trivial identity $\xi^{\rightarrow} \equiv \xi^{\leftarrow}$.

At the isotopic level we have the same occurrence. In fact, the isoantisymmetric algebra $\hat{\xi}^{-}$ with product $A\hat{\times}B - B\hat{\times}A$ is also made up of two contributions, the first for the isomultiplication to the right $\hat{\xi}^{\rightarrow}$: $A\rightarrow B = A\hat{\times}B$ and the second for the isomultiplication to the left $\hat{\xi}^{\leftarrow}$: $B\leftarrow A = B\times A$ with Lie–Santilli product

$$[A, B\hat{]} = A \rightarrow B - B \leftarrow A \equiv A \hat{\times} B - B \hat{\times} A, \tag{1.1.17}$$

in which case we also have the identity $\hat{\xi}^{\rightarrow} \equiv \hat{\xi}^{\leftarrow}$.

manifestly better than that permitted by the isotopies.

The name *genotopies* was suggested by Mrs. Carla Santilli and submitted in [6a] from the Greek meaning of *inducing configuration* and interpreted as inducing covering theories. As we shall see, the lifting of the unit $I \to \hat{1}^{>} = \hat{R}^{-1}$ while jointly lifting the product by the *inverse* amount, $A \times B \to A > B = A \times \hat{R} \times B$, permits the preservation of the original axioms at the abstract level. The genotopies therefore verify in a natural way *all* conditions 1)–5) of the preceding subsection, including condition 5), evidently when properly treated, as we shall see.

To understand the chain of generalized methods (1.1.1) it is important to point out since these introductory words that *the mathematical origin of the genotopies is the ordering of the product and the differentiation of the product to the right* $A>B$ *from that to the left* $A<B$, $A>B \neq A<B$.

This occurrence can be first illustrated with numbers and then with the structure of the time evolution law. Let $R(n,+,\times)$ be the ordinary field of real numbers n with sum + and multiplication ×. Owing to millennia of use, we are accustomed to using without distinction either the "multiplication of 2 time 3 to the right", $2 \to 3$, or "the multiplication of 3 time 2 to the left", $2 \leftarrow 3$ because the results are trivially the same., $2 \to 3 \equiv 2 \leftarrow 3 \equiv 2 \times 3 = 6$, and the product is commutative, $2 \to 3 \equiv 3 \to 2 \equiv 2 \leftarrow 3 \equiv 3 \leftarrow 2$.

Under isotopic lifting the above scenario persists. In fact, we have the "isotopic multiplication of 2 time 3 to the right", $2 \to \hat{T} \to 3$, and the "isotopic multiplication of 3 time 2 to the left", $2 \to \hat{T} \to 3$. The results are also the same because of the assumed Hermiticity of $\hat{T}$, $2 \to \hat{T} \to 3 \equiv 2 \leftarrow \hat{T} \leftarrow 3$, and the isotopic product is also commutative, $2 \to \hat{T} \to 3 \equiv 3 \to \hat{T} \to 2 \equiv 2 \leftarrow \hat{T} \leftarrow 3 \equiv 3 \leftarrow \hat{T} \leftarrow 2 = 2 \times \hat{T} \times 3$.

When passing to the more general genotopic level, the ordering appears in its full nontrivial light, because the genoproduct remains commutative, $2>3 \equiv 2 \times \hat{R} \times 2 \equiv 3>2 = 3 \times \hat{R} \times 2$, but the values of the two ordered products are now different, $2>3 = 2 \times \hat{R} \times 3 \neq 2<3 = 2 \times^{\hat{S}} \times 3$, because $\hat{R} \neq \hat{S}$.

The fundamental property identified by Santilli (see the more recent study [15]) is that the axioms of a field are preserved even under the selection of one given ordering of the multiplication, that is, the sets $R(n,+,\to)$ and $R(n,+,\leftarrow)$, where

As we shall see, a number of relatively "simple" biological structures can be described with the isotopic methods, while more complex structure require the more complex genotopic methods.

But quantitative sciences will never admit final theories. The third and final class of generalized methods considered in this memoir is that of the *hyperstructures* in the form recently introduced by Santilli and Vougiouklis [16] the first day of the International Workshops held at the Institute for Basic Research in Molise in August 1995, and they will be referred to as the *Santilli–Vougiouklis hyperstructures.*

Note that the isounit has one single value and the same occurs for the genounits for each ordering. *The main idea of the hyperstructures here considered is the assumption of a generalized unit which is a finite or infinite and ordered or non–ordered set,* and we shall write

$$I \to \{\hat{1}^{>}\} = \{\hat{1}^{>}_1, \hat{1}^{>}_2, \hat{1}^{>}_3, ...\},$$

$$I \to \{^{<}\hat{1}\} = \{^{<}\hat{1}_1, ^{<}\hat{1}_2, ^{<}\hat{1}_3, ...\} \qquad (1.1.22)$$

with interconnecting conjugation

$$\{\hat{1}^{>}\} = \{^{<}\hat{1}\}^{\dagger}. \qquad (1.1.23)$$

as well as the *dual* lifting of the product

$$A \times B \to A > B = A \times \{\hat{R}\} \times B = \{A \times \hat{R}_1 \times B,\ A \times \hat{R}_2 \times B, ...\},$$

$$A \times B \to A < B = A \times \{\hat{S}\} \times B = \{A \times \hat{S}_1 \times B, A \times \hat{S}_2 \times B, ...\}, \qquad (1.1.24)$$

under the conditions (referred to the individual terms of each set)

$$\{\hat{1}^{>}\} = \{\hat{R}\}^{-1}: \quad \{\hat{1}^{>}\} > A \equiv A = A > \{\hat{1}^{>}\},$$

Hyperstructures are generally defined *without* a unit (see, e.g., ref.s [17–19]), thus without realistic possibility of applications to actual measurements (which evidently require a basic unit for its very conception). The hyperstructures considered in this memoir, those with a well defined left and right hyperunit, were submitted for the first time in ref. [16] and are developed further in this memoir.

It is also evident that the above hyperstructures contain as particular cases the genotopies which, in turn, contain as particular cases the isotopies which, in turn, contain as particular case the conventional mathematical methods, according to a sequence representable via the inclusive chain of units

$$I \subset \hat{1} \subset (\hat{1}^{>}, {}^{<}\hat{1}) \subset (\{\hat{1}^{>}\}, \{{}^{<}\hat{1}\}). \tag{1.1.28}$$

with corresponding enclosure properties for the (strong) multiplications.

This perspective illustrates the fact that genotopies can be re–interpreted as the simplest possible realization of the Santilli–Vougiouklis hyperstructures.

To state it in a nut shell, the entire content of this memoir is reducible to one single notion: the generalization of the basic unit I, which has remained unchanged since biblical times, into forms of progressively increasing complexities as required by the complexity of the biological systems to be represented. The memoir merely deals with the identification of the methods for the mathematically consistent handling of such basic notion.

Our task will essentially be that of identifying the basic methods for the correct handling of the new methods. By recalling that conventional methods include analytic, algebraic, geometric and other profiles, the study of all necessary aspects of the new methods will evidently require several different monographs and a considerable number of years to write them.

In a situation of this type, we have to make a selection. In this memoir we shall identify only those aspects which are essential for correct applications, with particular reference to really fundamental notions, that is, the generalized numbers, angles, elementary functions, differential calculus and vector spaces. We shall then focus our attention on the emerging chain of generalized, classical

2: ELEMENTS OF ISOTOPIC METHODS

2.1: Mathematical foundations

2.1.A: Kadeisvili classification

All contemporary quantitative sciences are based on the most elementary possible unit, the number +1, which has remained unchanged since biblical times.

As indicated in Sect. 1, the main idea of the isotopies is the lifting of the trivial unit I of a conventional theory into a nowhere singular, symmetric, real–valued, Hermitean and N–dimensional matrix $\hat{1} = (\hat{1}_i^j) = (\hat{1}_i^j) = \hat{1}^{-1} = (T_i^j)^{-1} = (T_i^j)^{-1}$, i,j 1, 2, ..., N, called the *isounit*, whose elements have a smooth but otherwise arbitrary functional dependence on the local coordinates x, their derivatives $\dot{x}$, $\ddot{x}$, ..., with respect to an independent variable t and any needed additional local quantity, such as local temperature τ, the local density μ, the local index of refraction n (as well as possible quantum mechanical quantities studied later on in this section)

$$I \rightarrow \hat{1}(x, \dot{x}, \ddot{x}, \tau, \mu, n, ...) . \qquad (2.1.1)$$

The original theory is then reconstructed in such a way to admit $\hat{1}$ as the new left and right unit, as outlined in this section. This requires for consistency the lifting of the totality of the mathematical structure of the original theory, including fields, metric spaces, functional analysis, algebras, groups, geometries, etc. The new theory is called an isotopic image of the old because the two

therefore indicate from these preliminary elements the apparent possibility that *under certain specifications, biological structures can master the direction of time.* One of the objectives of the isotopies is that achieving a quantitative representation of this prediction for its future experimental resolution.

The isotopies of Class IV contain a singularity in the unit by conception and, as such, they are particularly delicate to handle on mathematical grounds. This presentation shall therefore be restricted hereon to the isotopies of Class III and consider the possible zeros of the isounit as a separate particular case.

Unless otherwise specified, all isounits and isotopic elements are hereon assumed to be of Kadeisvili Class III.

2.1.B: Isofields

The fundamental isotopies are those of fields. Let $F = F(a,+,\times)$ be a field (hereon assumed to have characteristic zero) with elements a, b, ..., sum $a + b$, multiplication $a\times b := ab$, additive unit 0, multiplicative unit 1, and familiar properties $a + 0 = 0 + a = a$, $a\times 1 = 1\times a = a$, $\forall\ a \in F$, and others. We have in particular: the field $R(n,+,\times)$ of real numbers n, the field $C(c,+,\times)$ of complex numbers c, and the field $Q(q,+,\times)$ of quaternions q.

Definition 2.1 [15]: *An "isofield"* $\hat{F} = \hat{F}(\hat{a},+,\hat{\times})$ *is a ring with elements* $\hat{a} = a\times\hat{1}$, *called "isonumbers", where* $a \in F$, *and* $\hat{1}$ *is a Class III isotopic element generally outside* F, *equipped with two operations* $(+, \hat{\times})$, *where* $+$ *is the conventional sum of* F *with conventional additive unit* 0, *and* $\hat{\times}$ *is a new multiplication*

$$\hat{a}\,\hat{\times}\,\hat{b} := \hat{a}\times\hat{T}\times\hat{b},\quad \hat{1} = \hat{T}^{-1}, \qquad (2.1.2)$$

called "isomultiplication", which is such that $\hat{1}$ *is the left and right unit of* $\hat{F}$,

We therefore have the isofield $\hat{R}(\hat{n},+,\hat{\times})$ of *isoreal numbers* $\hat{n}$; the isofield $\hat{C}(\hat{c},+,\hat{\times})$ of *isocomplex isonumbers* $\hat{c}$; and the isofield $\hat{Q}(\hat{q},+,\hat{\times})$ of *isoquaternions* $\hat{q}$ (see [18] for the *isooctonions*). Since $\hat{F}$ preserves by construction all axioms of F, it is called an *isotope* of F and the lifting $F \to \hat{F}$ is called an *isotopy*. All conventional operations dependent on the multiplication on F are generalized on $\hat{F}(\hat{a},+,\hat{\times})$, thus yielding isotopies of powers, quotients, square roots, etc. These isotopic operations are however such that $\hat{1}$ preserves all the original axiomatic properties of I, i.e., $\hat{1}^{\hat{n}} = \hat{1}*\hat{1}*...*\hat{1}$ (n–times) $= \hat{1}$, $\hat{1}^{\hat{\frac{1}{2}}} = \hat{1}$, $\hat{1}\hat{/}\hat{1} = \hat{1}$, etc. (see [18] for details).

Note that the isotopy is restricted to the sum, as indicated by the symbol $\hat{F}(\hat{a},+,\hat{\times})$, because the lifting of a field into the form $\hat{F}(\hat{a},\hat{+},\hat{\times})$ inclusive of the lifting of the sum, such as $+ \to \hat{+} = + K +$ with corresponding lifting of the additive unit $0 \to \hat{0} = -K$, $K > 0$, $K \in F$, generally implies the loss of the original axioms, such as the loss of closure (1.4). Therefore, the lifting of the sum is not an isotopy. Moreover, quantities which are conventionally finite on $F(a,+,\times)$ as well as on $\hat{F}(\hat{a},+,\hat{\times})$, such as the exponentiation on F, $e^a = I + a/1! + a^2/2! + ...$, or that on $\hat{F}$, $\hat{e}^{\hat{a}} = \hat{1} + a/1! + a\hat{\times}a/2! + ... = \{e^{a\times\hat{T}}\}\hat{1} = \hat{1}\{e^{\hat{T}\times a}\}$, become divergent under the liftings $+ \to \hat{+} = + K +$, $0 \to \hat{0} = -K$, $K \in F$ [18]. For this reason only the isotopies of the multiplication are used in applications at this writing [15].

Despite its simplicity, the lifting $F \to \hat{F}$ has significant implications in number theory itself. For instance, real numbers which are conventionally prime (under the tacit assumption of the unit 1) are not necessarily prime with respect to a different unit [15]. This illustrates that most of the properties and theorems of the contemporary number theory are dependent on the assumed unit and, as such, admit intriguing isotopies. Also, the isotopies permit the conception of a new generation of cryptograms which are expected to be difficult to break because of the availability of an infinite number of different units which are not admitted by the conventional number theory.

It is important to understand that an isofield of Class III, $\hat{F}_{III}(\hat{a},+,\hat{\times})$ is the unit of two isofields, one of Class I $\hat{F}_{I}(\hat{a},+,\hat{\times})$ in which the unit is positive–definite and one of class III $\hat{F}_{II}(\hat{a},+,\hat{\times})$ in which the isounit is negative–definite,

field $F(a,+,\times)$ as well as a new field, the *isodual field* $F^d(a^d,+,\times^d)$ with *isodual unit* $I^d = -I$, *isodual numbers* a^d, where a represents ordinary numbers, and isodual multiplication $\times^d = -\times$.

To prevent misrepresentations of subsequent sections the reader is advised to get acquainted with the latter new numbers, such as the *isodual real field* $R^d(n^d,+,\times^d)$ of *isodual real numbers* $n^d = -n$, $n \in R$. Since n is an arbitrary real number, the set of all possible isodual numbers n^d *coincides* with the set of ordinary numbers. Thus, the distinction occurs in the units and operations.

We finally note that the lifting of the unit and related multiplication require corresponding liftings of all operations defined on them, such as, *isosquare, isoquare root, isoquotient, etc,*

$$\hat{a}^{\hat{2}} = \hat{a}\hat{\times}\hat{a} = (a \times a) \times \hat{1}, \quad \hat{A}^{\hat{\frac{1}{2}}} = a^{\frac{1}{2}} \times \hat{1}^{\hat{\frac{1}{2}}}, \quad \hat{7} = 7 \times \hat{1}, \text{ etc.} \qquad (2.1.9)$$

It is then easy to see that the isounit verifies all axiomatic properties of the conventional unit, e.g.,

$$\hat{1}^{\hat{n}} = \hat{1}\hat{\times}\hat{1}\hat{\times}...\hat{\times}\hat{1} \equiv \hat{1}, \quad \hat{1}^{\hat{\frac{1}{2}}} \equiv \hat{1}, \quad \hat{1}\hat{/}\hat{1} \equiv \hat{1}, \text{ etc.} \qquad (2.1.10)$$

For additional technical studies we refer the reader to ref.s [9,11,15].

It may be of future guidance to introduce since these introductory lines a few examples of isounits used in applications. One of the simplest possible example is *the use of the isounit for the representation of extended, nonspherical and deformable shapes.* For instance, a spheroidal ellipsoids in three dimension can be represented via the isounit

$$\hat{1} = \text{diag.}\,(n_1^2, n_2^2, n_3^2), \qquad (2.1.11)$$

where the quantities n_k^2 are sufficiently well behaved, real valued and positive-definite functions of local quantities, such as the intensity of external fields, the local pressure, etc. As we shall see shortly, the isotopies of the Euclidean space with the above isounit will indeed characterize all infinitely possible ellipsoids.

with anything *except* the Hamiltonian. Without any claim of being unique, contact interactions are represented in isotopic methods via the isounit.

In general, the isounit used in application is a matrix with the dimension of the used carrier space (two–, three– and four–dimensions for problems in the plane, space and space–time, respectively) which is generally *nondiagonal* whose elements have a local–differential as well as nonlocal integral dependence on local physical quantities.

The notion of isonumbers was presented, apparently for the first time, by this author at the conference *Differential Geometric Methods in Mathematical Physics*, held at the University of Clausthal, Germany, in 1980. The first mathematical treatment appeared in ref. [26] of 1982. A systematic study is available in above quoted ref. [15], while additional studies and applications are presented in monographs [9,11].

2.1.C: Isospaces

The mathematical and physically most important consequences of isofields are that they imply, for evident consistency, corresponding isotopies of *all* quantities defined over conventional fields. Let $E(x,\delta,R)$ be an N–dimensional Euclidean space, with local chart $x = \{x^k\}$, $k = 1, 2, ..., N$, N–dimensional metric $\delta = \text{diag.}\,(1, 1, ..., 1)$ and invariant separation between two points $x, y \in E$,

$$(x - y)^2 = (x^i - y^i)\,\delta_{ij}\,(x^j - y^j) \in R(n,+,\times), \qquad (2.1.15)$$

over the reals $R(n,+,\times)$, where the convention on the sum of repeated indices is assumed hereon .

Definition 2.2 [27]: *An "isoeuclidean space"* $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ *is an* N–*dimensional metric space defined over an isoreal isofield of Class III* $\hat{R}(\hat{n},+,\hat{\times})$ *with an* N×N–*dimensional isounit* $\hat{1}$, *equipped with the "isometric"*

$$\hat{\delta} = (\hat{\delta}_{ij}\hat{}) = \hat{T} \times \delta, \qquad \hat{1} = \hat{T}^{-1}, \qquad (2.1.16)$$

Because of the above occurrences, isospaces can be practically treated via the conventional coordinates x^k rather than the isotopic ones $\hat{x}^k = x^l \hat{\times} \hat{1}$ whenever no confusion arises.

Note that the coordinates of E and $\hat{E}$ coincide in their contravariant form, but not in their covariant form, for which $x_k = \delta_{ki} x^i$ and $x_k = \hat{\delta}_{ki} x^i$. Because of the latter occurrence, the symbol x will be used for the coordinates of conventional spaces, while the symbol $\hat{x}$ will be used for the coordinates of isospaces. When writing $\hat{\delta}(x, ...)$ we refer to the *projection* of the isometric $\hat{\delta}$ in the original space.

Despite its simplicity, the lifting $E(x,\delta,R) \to \hat{E}(\hat{x},\hat{\delta},\hat{R})$ also has significant implications. In fact, the functional dependence of the isounit $\hat{1}$ remains unrestricted under isotopies. The isometric $\hat{\delta}$ therefore has the same dependence of $\hat{1}$ and $\hat{T}$, $\hat{\delta} = \hat{\delta}(x, \dot{x}, \ddot{x}, \psi, \mu, n, ...)$. The isoseparation (2.10) is therefore the most general possible integro-differential separation with signature (+, +, +).

Isogeometries have novel properties which do not appear to have propagated as yet into the mathematical literature. For instance, the conventional trigonometry on the two-dimensional Euclidean space $E(x,\delta,R)$, δ = diag. (1, 1) (Gauss plane) is lost under lifting to a two-dimensional Riemannian space $\Re(x,g(x),R)$, but trigonometry can be reformulated in the two-dimensional isospace $\hat{E}(\hat{x},\hat{\delta}(x,\dot{x},\ddot{x}...),\hat{R})$ resulting in the so-called *isotrigonometry* reviewed in App. A (see [11], App. 5.C, for additional studies). An intriguing application is the formulation of the *Pythagorean theorem for a triangle with curved sides* (because for each given such triangle, there exists an isotopy such that its image in isospace is an ordinary triangle with rectilinear sides).

The *isominkowskian space* was introduced in paper [28] of 1993 and then studied in details in the more recent article [29], and will be studied later on in this memoir. The *isoriemannian spaces* and related geometry are studied in detail in monograph [11] and they will not be reviewed for brevity. We only mention that *all possible conventional Riemannian metrics* g(x) *are a trivial particular case of the isoeuclidean metric of the same dimension,* $g(x) \in \hat{\delta}(x, \dot{x}, \ddot{x}, \tau, \mu, n, ...)$. Even though the curved profile will not be studied for brevity, the reader should be aware that *isotopic representations of biological structures*

isoconvergent to $\hat{1}$ when

$$Lim_{k \to \infty} \hat{|} \hat{1}_k - \hat{1} \hat{|} = 0 , \qquad (2.1.22)$$

while the *isocauchy condition* can then be expressed by

$$\hat{|} \hat{1}_m - \hat{1}_n \hat{|} < \hat{\delta} = \delta \times \hat{1} , \qquad (2.1.23)$$

where δ is real and m and n are greater than a suitably chosen $N(\delta)$. The isotopies of other notions of continuity, limits, series, etc. can be easily constructed [26]. Note that functions which are conventionally continuous are also isocontinuous. Similarly, a series which is strongly convergent is also strongly isoconvergent.

However, *a series which is strongly isoconvergent is not necessarily strongly convergent* (ref. [19], p. 271). As a result, *a series which is conventionally divergent can be turned into a convergent form under a suitable isotopy*. This mathematically trivial property has rather important applications, e.g., for the reconstruction of convergence at the isotopic level.

The notion of an N–dimensional *isomanifold* was first studied by Tsagas and Sourlas [30,31]. In this paper we use the following simplest possible realization of isomanifolds. Since an N×N–dimensional isounit is positive–definite, it can always be diagonalized into the form

$$\hat{1} = diag. (b_1^{-2}, b_2^{-2}, ..., b_N^{-2}) > 0 , \quad b_k > 0 , k = 1, 2, ..., N , \qquad (2.1.24)$$

Consider then N isoreal isofields $\hat{R}_k(\hat{n},+,\hat{\times})$ each characterized by the isounit $\hat{1}_k = b_k^{-2}$ with (ordered) Cartesian product

$$\hat{R}^N = \hat{R}_1 \times \hat{R}_2 \times ... \times \hat{R}_N . \qquad (2.1.25)$$

Since $\hat{R}_k \approx R$, it is evident that $\hat{R}^N \approx R^N$, where R^N is the Cartesian product of N conventional fields $R(n,+,\times)$. But the total unit of $\hat{R}^N$ is expression (1.15). Therefore, one can introduce a topology on $\hat{R}^N$ via the simple isotopy of the conventional

$$\hat{1} = \hat{1}_1 \times \hat{1}_2 \times \ \ldots \ \times \hat{1}_N \, . \qquad (2.1.30)$$

In summary, Tsagas-Soulas isotopology includes the conventional local coordinate x(t) which represents the trajectory of the center-of-mass which is characterized by conventional local-differential (i.e., pootential) terms plus nonlocal-integral contributions.

For all additional aspects of isomanifolds and related topological properties we refer the interested reader to Tsagas and Sourlas [13,20]. It should be noted that their study is referred to $M(\hat{R}^N)$, rather than to $\hat{M}(\hat{R}^N)$ because of the use of the *conventional* topology τ (i.e. a topology with the conventional N×N-dimensional unit I). The extension to $\hat{M}(\hat{E})$ with the isotopology $\hat{\tau}$ is introduced here apparently for the first time.

The isotopies of functional analysis, called *isofunctional analysis,* were introduced by Kadeisvili [25] and cannot possibly be reviewed here for brevity (see App. A only for the most elementary isofunctions).

2.1.E: Isodifferential calculus

Let E(x,δ,R) be the ordinary N-dimensional Euclidean space with local coordinates $x = \{x^k\}$, k = 1, 2, ..., N, and metric δ = diag. (1, 1, 1) over the reals R(n,+,×). Let $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ be its isotopic image with local coordinates $\hat{x} = \{\hat{x}^k\}$ and isometric $\hat{\delta} = \hat{T}\delta$ over the isoreals $\hat{R}(\hat{n},+,\hat{\times})$. Let the isounit be given by the N×N matrix of Class III, $\hat{1} = (\hat{1}_i{}^j) = (1_i{}^j) = \hat{T}^{-1} = (T_i{}^j)^{-1} = (T_i{}^j)^{-1}$ whose elements have a smooth but otherwise arbitrary functional dependence on the local coordinates, their derivatives with respect to an independent variable and any needed additional quantity, $\hat{1} = \hat{1}(\hat{x},...)$. The following properties then hold from Definition 2.2:

$$\hat{x}^k \equiv x^k \, , \hat{x}_k = \hat{\delta}_{ki} \hat{x}^i = \hat{T}_k{}^i \delta_{ij} \hat{x}^j = \hat{T}_k{}^i \delta_{ij} x^j = \hat{T}_k{}^i x_i \, , \ x_i = \delta_{ij} x^j,$$

$$\hat{x}^i \hat{\delta}_{ij} \hat{x}^j = \hat{x}^i \hat{T}_i{}^j \delta_{jm} \hat{x}^m = \hat{x}_i \hat{\delta}^{ij} \hat{x}_j \equiv \hat{x}^k \hat{x}_k = \hat{x}_k \hat{x}^k \, , \hat{\delta}^{ij} = [\, (\hat{\delta}_{mn})^{-1} \,]^{ij} \, ,$$

(2.1.34)

The above definition and the axiom-preserving character of the isotopies then permit the lifting of the various aspects of the conventional differential calculus. We here mention for brevity the following isotopies: the *isodifferentials of an isofunction* of contravariant (covariant) coordinates $\hat{x}^k$ ($\hat{x}_k$) on $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ are defined via the isoderivatives according to the respective rules

$$\hat{d}\,\hat{f}(\hat{x})|_{contrav.} = \frac{\hat{\partial}\hat{f}}{\hat{\partial}\hat{x}^k}\,\hat{d}\hat{x}^k = \hat{T}_k{}^i\,\frac{\partial f}{\partial x^i}\,\hat{1}^k{}_j\,dx^j = df(x)\ ,$$

$$\hat{d}\,f(x)|_{covar.} = \frac{\hat{\partial}\hat{f}}{\hat{\partial}\hat{x}_k}\,\hat{d}\hat{x}_k = \hat{1}^k{}_i\,\frac{\partial f}{\partial x_i}\,\hat{T}_k{}^j\,dx_j = df(x)\,; \qquad (2.1.35)$$

an iteration of the notion of isoderivative leads to the *second-order isoderivatives*

$$\frac{\hat{\partial}^2\,\hat{f}(\hat{x})}{\hat{\partial}\,x^{k\,2}} = \hat{T}_k{}^i\,\hat{T}_k{}^j\,\frac{\partial^2 f(x)}{\partial x^i\,\partial x^j}\ ,\qquad \frac{\hat{\partial}^2\,\hat{f}(\hat{x})}{\hat{\partial}\,x_k{}^2} = \hat{1}^k{}_i\,\hat{1}^k{}_j\,\frac{\partial^2 f(x)}{\partial x_i\,\partial x_j}\ \text{(no sums on k}$$

(2.1.36)

and similarly for isoderivatives of higher order; the *isolaplacian* on $\hat{E}(\hat{x},\hat{\delta},\hat{R})$ is given by

$$\hat{\Delta} = \hat{\partial}_k\,\hat{\partial}^k = \hat{\partial}^i\,\hat{\delta}_{ij}\,\hat{\partial}^j = \hat{\partial}^i\,\delta_{ij}\,\partial^j = \hat{1}^i{}_k\,\partial^k\,\delta_{ij}\,\partial^j\ ,\ \ \hat{\partial}_k = \hat{\partial}/\hat{\partial}\hat{x}^k,\ \partial_k = \partial/\partial x^k,\ \text{etc.}\ ,$$

(2.1.37)

and results to be different than the corresponding expression on a Riemannian space $\Re(x,g,R)$ with metric $g(x) = \hat{\delta}$, $\Delta = \hat{\delta}^{-1/2}\,\partial_i\,\hat{\delta}^{1/2}\,\hat{\delta}^{ij}\,\partial_j$.

A few examples are in order. First note the following properties derived from definitions (2.3) and (2.4),

$$\hat{\partial}\hat{x}^i\,/\,\hat{\partial}\hat{x}^j = \delta^i{}_j\ ,\ \ \hat{\partial}\hat{x}_i\,/\,\hat{\partial}\hat{x}_j = \delta_i{}^j\ ,\ \ \hat{\partial}\hat{x}_i\,/\,\hat{\partial}\hat{x}^j = \hat{T}_i{}^j,\quad \hat{\partial}\hat{x}^i\,/\,\hat{\partial}\hat{x}_j = \hat{1}^i{}_j\ . \qquad (2.1.38)$$

Next, we have the simple isoderivatives

have

$$\hat{\partial}_t \hat{\partial}_k \hat{f}(t, \hat{x}) = \hat{\partial}_t [\hat{\partial}_k \hat{f}(t, \hat{x})] = \hat{\partial}_t [T_k^{\ i}(t, x, ...) \partial_i f(t, x)] . \qquad (2.1.42)$$

Additional properties of the isodifferential calculus will be identified during the course of our analysis.

Note also that the ordinary differential calculus is local–differential on M(E). The isodifferential calculus is instead local–differential on $\hat{M}(\hat{E})$ but, when projected on M(E), it becomes *integro–differential* because it incorporates integral terms in the isounit.

2.2: Isotopies of differential geometries

2.2.A: Introduction.

We are now equipped to study the central methodological tool of this memoir, the isotopies of the Euclidean geometry. For clarity, in this section we shall study first the isotopies of of Class I (which preserve the signature (+,+,+) of the conventional geometry), then those of Class II (which changes the signature into (–, –, –)) and finally we shall combine them together into Class III (for which the signature is arbitrary).

The geometric isotopies here studied were introduced by this author [28] in 1983 under the name of *Euclidean–isotopic geometry,* or *isoeuclidean geometry* for short, as a particular case of the *isominkowskian geometry* outlined in App. B. Subsequent studies have indicated that there is the emergence of a *new geometry* because the isotopies preserve the original axioms of the *flat* Euclidean geometry, but also embody at the same time *curvature* and other features belonging to *different* geometries.

The above main results can be anticipated from these introductory lines. In fact, the isotopies preserve by assumption the original geometric axioms, and therefore permit the preservation of the conventional features of the Euclidean

approach by Descartes.

Consider the conventional three–dimensional *Euclidean vector space* $V(r,\odot,R(n,+,\times)$ with elements r (vectors), their composition $r\odot r'$ (scalar product) over the field R of real numbers n equipped with the conventional addition + and multiplication × and respective additive unit 0 and multiplicative unit I.

Our first objective is to reconstruct $V(r,\odot,R)$ under isotopies, that is, when defined over an isofield $\hat{R}(\hat{n},+,\hat{\times})$ of isonumbers $\hat{n} = n\times\hat{I}$, equipped with the isosum $\hat{n} + \hat{n}' = (n+n')\times\hat{I}$ with isomultiplication $\hat{n}\hat{\times}\hat{n}' = \hat{n}\times\hat{T}\times\hat{n}' = (n\times n')\times\hat{I}$, equipped with the conventional additive unit $\hat{0} \equiv 0$ and a multiplicative isounit $\hat{1} = \hat{T}^{-1}$ which is a positive–definite quantity *outside the original field* R (e.g., $\hat{1}$ is an integral).

Let us begin with the study of the isotopies of the line.

Definition 2.5: *An "isoline" is the image of the ordinary line on the reals under the lifting* $R(n+,\times) \to \hat{R}(\hat{n},+,\hat{\times})$.

Coordinates on the isoline can be introduced as in the ordinary case, although they are now isonumbers, that is, ordinary numbers multiplied by the isounits,

$$\hat{x} = x\times\hat{1}\,, \tag{2.2.1}$$

on the isofield $\hat{R}$, and are thus called *isocoordinates*. One can first set up the *isoorigin* $\hat{0} = 0\times\hat{1}$. Then the *isopoint* on the isoline are arbitrary, positive or negative isonumbers $\hat{x}$. The isodistance among two isopoints is given by the isonorm on $\hat{R}$ (Ch. I.2)

$$\hat{D} = \hat{|}(\hat{x}-\hat{x}')\hat{|} = [(x-x')\times\hat{T}\times(x-x')]^{1/2}\times\hat{1}, \tag{2.2.2}$$

and, as such, it is an isonumber.

One of the important implications of the isotopies of the straight line is that, even though the axioms are the same, the values of the distance among two points is different for lines and isolines with the same points x and x′. In fact, $\hat{D}/D \neq I$ and $\neq\hat{1}$.

$y\hat{\times}\hat{1}$, $z\hat{\times}\hat{1}$) *on isospaces* $\hat{V}(\hat{r},+,\hat{\odot},\hat{R}(\hat{n},+,\hat{\times}))$ *over the isofield* $\hat{R}(\hat{n},+,\hat{\times})$ *with isounit* $\hat{1} > 0$ of Class I *(Sect. I.2.2) equipped with the original sum* + *and an isoproduct* $\hat{\odot} = \odot\hat{T}\odot$, $\hat{1} = \hat{T}^{-1} = \hat{1}^{\dagger} > \hat{0}$, *verifying the following properties for all possible* $\hat{r}, \hat{r}' \in \hat{V}$ and $\hat{n}, \hat{n}', \hat{n}'' \in \hat{R}$:

1) $\hat{r} + \hat{r}' = \hat{r}' + \hat{r}$,

2) $(\hat{r} + \hat{r}') + \hat{r}'' = \hat{r} + (\hat{r}' + \hat{r}'')$,

3) *the set* $\hat{V}$ *includes the element* $\hat{0}$ *such that* $\hat{r} + \hat{0} \equiv \hat{r}$,

4) *for every element* $\hat{r}$ *there is an element* $-\hat{r}$ *such that* $\hat{r} + (-\hat{r}) = \hat{0}$,

5) $(\hat{n} + \hat{n}')\hat{\times}\hat{r} = \hat{n}\hat{\times}\hat{r} + \hat{n}'\hat{\times}\hat{r}$,

6) $\hat{n}\hat{\times}(\hat{r} + \hat{r}') = \hat{n}\hat{\times}\hat{r} + \hat{n}\hat{\times}\hat{r}'$,

7) $\hat{n}\hat{\times}\hat{n}'\hat{\times}\hat{r} = (\hat{n}\hat{\times}\hat{n}')\hat{\times}\hat{r} = \hat{n}\hat{\times}(\hat{n}'\hat{\times}\hat{r})$,

8) $\hat{1}\hat{\times}\hat{r} = \hat{r}\hat{\times}\hat{1} \equiv \hat{r}$,

9) *the isoproduct is an isonumber, i.e.,* $\hat{r}\hat{\odot}\hat{r}' = \hat{n} = n\hat{\times}\hat{1} \in \hat{R}$,

10) $(\hat{n}\hat{\times}\hat{r})\hat{\odot}\hat{r}' = \hat{n}\hat{\times}(\hat{r}\hat{\odot}\hat{r}')$

11) $\hat{r}\hat{\odot}(\hat{r}' + \hat{r}'') = \hat{r}\hat{\odot}\hat{r}' + \hat{r}\hat{\odot}\hat{r}''$,

12) $\hat{r}\hat{\odot}\hat{r} = \hat{0}$ iff $\hat{r} = 0$,

13) $\hat{r}\hat{\odot}\hat{r}' \leq \hat{r}\hat{\odot}\hat{r}'' + \hat{r}''\hat{\odot}\hat{r}'$,

14) $\hat{r}\hat{\odot}\hat{r}' \neq \hat{r}\hat{\times}\hat{r}'$.

The "isoeuclidean metric space", or "isoeuclidean space" for short, is the isospace $\hat{E}(\hat{r},+,\hat{\odot},\hat{R}(n,+,\times))$ *equipped with the "isodistance"*

$$\hat{D} = (\hat{r}\hat{\odot}\hat{r}')^{\hat{\frac{1}{2}}} = (r\odot T\odot r')^{1/2}\times\hat{1} \in \hat{R} . \qquad (2.2.3)$$

The "isoeuclidean geometry" of Class I is the geometry of the isoeuclidean spaces. Unless explicitly stated, the terms "isoeuclidean geometry' are specifically referred to the isogeometries of Classes I.

As one can see, Hamilton's original conception of "vectors" is merely reinterpreted as *isovectors*, that is, vectors belonging to a space in which the original scalar product is deformed by a given amount, $\odot \to \hat{\odot} = \odot\hat{T}\odot$, where $\hat{T}$ is fixed for all possible r, while jointly the basic unit is deformed by an amount which is the *inverse* of the deformation of the scalar product, $I \to \hat{1} = \hat{T}^{-1}$. As we

$$I_k = +1, \qquad k = x, y, z. \tag{2.2.6}$$

On the contrary, the isoeuclidean spaces have the infinite family of generally different isounits $\hat{1} = \text{diag.}\,(b_1^{-2}, b_2^{-2})$ which are the isounits of the basic $S\hat{O}(3)$ symmetry (see Vol. II), and which implies *infinitely many possible, dimensionless units for each axes which are different among themselves and different than* +1,

$$\hat{1}_k = b_k^{-2}, \qquad \hat{1}_k \neq \hat{1}_y \neq \hat{1}_z \neq +1. \tag{2.2.7}$$

The above differences have a number of intriguing mathematical implications studied below and applications studied in the final section. To begin, the Euclidean "space" is unique per each dimension, while there exist infinitely many possible isoeuclidean "spaces" per each dimension, although they all admit a single and unique abstract treatment for the same class.

By recalling from Sect. 2.C that

$$\hat{r}^{\hat{2}} = \hat{r} \hat{\times} \hat{r} = r_k \times \hat{1} \times \hat{1} \times r^k \times \hat{1} = (r_k \times r^k) \times \hat{1} =$$

$$= (r_k \times r^k) \times \hat{1} = (r^i \times \delta_{ij} \times r^j) \times \hat{1}, \tag{2.2.8}$$

we can ignore the isoscalar character of $\hat{X} = x \times \hat{1}$ and use only the coordinates x.

We mention that the isometric $\hat{\delta}$ could also be written as an 3×3 *isomatrix,* that is, *a* 3×3 *matrix whose elements are isoscalars*

$$\hat{\delta}_{ij} = \hat{1}_i^k \times \delta_{kj} \times \hat{1}. \tag{2.2.9}$$

In this case however the product of its elements among themselves and with any other quantity must be isotopic, thus reproducing again the fundamental isoinvariant (5.2.5e). For this reason, the isometric elements will be hereon considered to be ordinary scalars $\hat{\delta}_{ij}$ and their product with any quantity Q, $\hat{\delta}_{ij} \times Q$, an ordinary product.

Note that *the isoseparation coincides with the conventional separation for all possible scalar forms of the isounit.* In fact, in this case we have

original points. Then

$$D_{12} > d_{12} \quad \text{for} \quad \det \hat{1} < 1 ,$$

$$D_{12} < d_{12} \quad \text{for} \quad \det \hat{1} > 1 . \qquad (2.2.12)$$

The above property has a number of intriguing implications. First, the *same object has different sizes and shapes in the Euclidean and isoeuclidean geometries*, as illustrated in Fig. 2.1.

THE ISOBOX

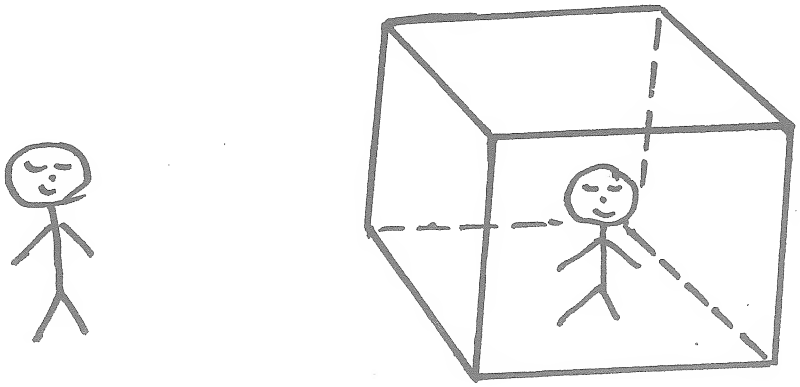


FIGURE 2.1. Consider a cube with sides of a given length d which is inspected from an outside observer in Euclidean space with geometric unit $I = \text{diag.}\ (1, 1, 1)$, and individual units per each axis $I_k = +1$, $k = x, y, z$. Suppose that there is a second observer in the interior of the cube belonging to an isoeuclidean space with isounit $\hat{1} = \text{diag.}\ (b_1^{-2}, b_2^{-2}, b_3^{-2})$, and individual isounits per each axis $\hat{1}_k = b_k^{-2}$, $k = z, y, z$. It is easy to see that the *same* object can have for the interior: 1) a volume arbitrarily *smaller* or *bigger* than d^3 depending on whether $\det \hat{1} > 1$ or < 1, respectively (Proposition 5.2.1) ; 2) a shape *different* than a cube; as well as 3) a

$$\hat{z} - \hat{z}_1 - \hat{p} \hat{\times} \hat{a}_3 = (z - z_1 - p\, a_3) \times \hat{1} = 0, \qquad (2.2.14)$$

where $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d} \in \hat{R}$, a, b, c, $d \in R$, p is an (ordinary) real parameter, and at least one of the isonumbers $\hat{a}$, $\hat{b}$ and $\hat{c}$ is not null. The isoline is called isostraight because its derivatives are constant.

Notice the importance for the consistency of the isoeuclidean geometry that the isocoordinates are isoscalars, i.e., are elements $\hat{x} = x \times \hat{1}$, $\hat{y} = y \times \hat{1}$, $\hat{z} = z \times \hat{1}$ of the isofield $\hat{R}$. In fact, the use for isocoordinates of of conventional scalars x, y, z would prohibit a consistent definition of isostraight isoline.

In isovector notation, the isoline can be represented by

$$\hat{r}_k - \hat{r}_{1k} + \hat{n} \hat{\times} \hat{a} = (r_k - r_{1k} - n\, a_k) \times \hat{1}, \quad k = 1, 2, 3, \qquad (2.2.15)$$

where $\hat{r} = \{\hat{r}^k\} = \{\hat{x}, \hat{y}, \hat{z}\}$, $\hat{r}_1$ is fixed point on the isoline, $\hat{a} = \{\hat{a}_1, \hat{a}_2, \hat{a}_3\}$ is the direction of the isoline itself and n is an arbitrary parameter.

GEOMETRIC PROPULSION

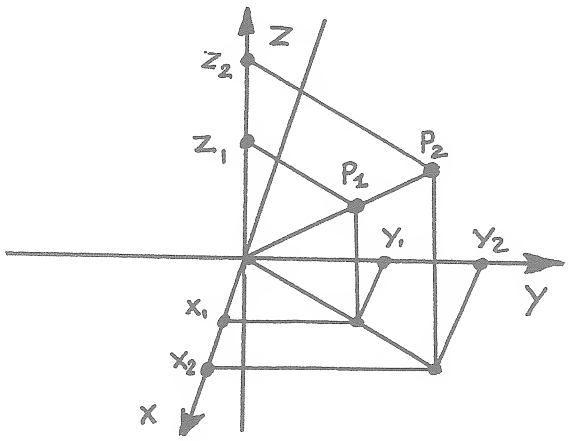


FIGURE 2.2: A schematic view of the *geometric propulsion* apparently introduced here for the first time. It is based on the idea of *realizing motion between two points via an isotopic alteration of the underlying geometry*, rather than the conventional displacement. Consider the Euclidean plane and two points P_1 and P_2

It then follows that the projection of the isostraight isoline (5.2.12a) in a flat space is given by

$$a \times b_1(t, r, ...) + b \times b_2(t, r, ...) + c \times b_3(t, r, ...) + d = 0 , \qquad (2.2.17)$$

which is evidently curved.

We therefore expect the existence of the inverse property, that is, given an arbitrary well behaved surface (5.2.15) in Euclidean space, there always exists an isotopy under which said surface is mapped into the isostraight isoline in isospace.

An *isopoint* in $\hat{V}(\hat{r},+,\hat{0},\hat{R})$ is a point $\hat{P}(\hat{x}, \hat{y}, \hat{z})$ with isocoordinates $\hat{x}$, $\hat{y}$, $\hat{z}$. Consider now two isovectors from the isoorigin $\hat{0}$ to the isopoints $\hat{P}_1$ and $\hat{P}_2$. An *isosegment* is the portion of an isostraight line between two isopoints.

In other conventional generalizations of the Euclidean metrics $\delta \to \hat{\delta}(t, r, \dot{r}, \ddot{r}, ...)$ the notion of angle is generally lost (as it is the case for the Riemannian geometry) because of the emergence of the curvature. A peculiarity of the isoeuclidean geometry is that, despite the most general possible functional dependence of the isometric, a generalized notion of angle can still be introduced. It is called the *isoangle,* denoted with the symbol $\hat{\alpha}$, and characterized by the expression on the isoplane $\hat{z} = \hat{0}$ for simplicity studied in detail in App. A

$$\text{isocos}\,\hat{\alpha} = \frac{x_1 b_1^2 x_2 + y_1 b_2^2 y_2}{(x_1 b_1^2 x_1 + y_1 b_2^2 y_1)^{\frac{1}{2}} (x_2 b_1^2 x_2 + y_2 b_2^2 y_2)^{\frac{1}{2}}} . \qquad (2.2.18)$$

which, as one can see, is an ordinary scalar (rather than an isoscalar) because the isounits cancels out in the ratio.

As studied in more details in Ch. II.6 of ref. [12] on the isorepresentation of Lie–Santilli isorotation group $\hat{O}(2)$, the explicit form of $\hat{\alpha}$ is given by

$$\hat{\alpha} = b_1 b_2 \alpha , \qquad (2.2.19)$$

where α is the original angle prior to the isotopies.[4] This implies that the

[4] We should indicate that the definition of isoangle for *nondiagonal* isometrics is

isopoint to another isopoint.

Isoaxiom II: *An isosegment can be prolonged continuously into an isostraight line from each end.*

Isoaxioms III: *For any given center and isoradius there is one and only one isosphere.*

Isoaxioms IV: *All isoright isoangles are equivalent.*

Isoaxioms V: *For each given isosegment between two isopoints there exist only two isoparallel lines, one per each isopoint, which are perpendicular to that isosegment.*

RECONSTRUCTION OF ANGLES IN THE ISOPLANE

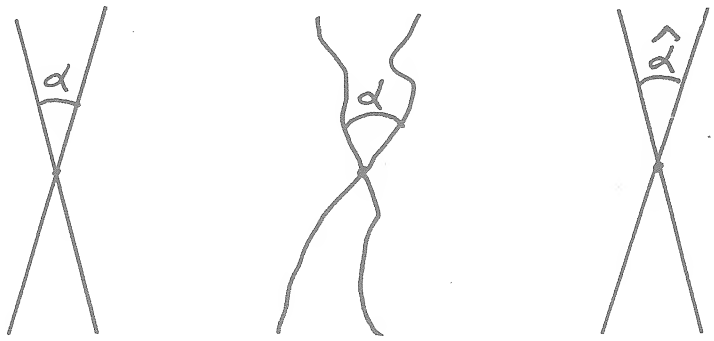


FIGURE 2.3. Diagram (A) depicts the origin of the notion of angle in the conventional Euclidean plane from two straight intersecting lines, which can be analytically expressed via the familiar expression

reduced from three to one dimension.

We learn in this way that *nondiagonal isotopies can reduce the number of effective dimensions of the original space.* This is the case of isoinvariant (5.2.19) for which the original dimension is three, with coordinates x, y, z, while the resulting dimension is one and represented by x, while the coordinates y and z remain outside the geometry.

One can see in this way additional peculiarities of the "isobox" of Fig. 2.1. In fact, the alteration of volume and shape of the cube and their variation in time for the internal observer should be complemented with the additional possibility that the number of dimensions themselves are changed in the interior. In fact, for isotopic element (2.65) the external observer perceives a three-dimensional cube, while the internal observer perceives it as a one-dimensional segment.

The invertible isotopies which alter the dimensionality of the original space are here called *degenerate Class I isotopies.* Rather than being a mere mathematical curiosity, the latter isotopies have emerge as having intriguing possibilities of applications studied in Vol.s II and III, such as a quantitative representation of the synthesis of neutrons from protons and electrons only as occurring in the interior of stars, which is possible under a nondiagonal Class I isotopy of the four-dimensional space-time of the electron down to two-dimensions.

Note that no study is available at this writing on the isoaxioms of the isoeuclidean geometry for nondiagonal isounits of Class I.

2.2.C: Basic properties of the isodual isoeuclidean geometry.

The isoeuclidean geometry of Class I studied above is suggested for the characterization of *motion forward in time.* In this subsection we study an antiautomorphic image of the above geometry which is appropriate for the separate characterization of *motion backward in time* in a causal way, which is called *isodual isoeuclidean geometry* and results to be of Class II. Subsequently, we shall combine the above two formulations in a single isogeometry of Class III.

corresponding classes. Unless explicitly stated, the terms "isodual isoeuclidean geometry" are referred to that of Class II.

The construction of the isodual geometry via map (5.2.21) is straighforward. the *isodual isostraight isoline* is the infinite set $\hat{R}^d(\hat{n}^d,+,\hat{x}^d)$ with *isodual isopoints* given by the elements $\hat{n}^d = n\times\hat{1}^d = -\hat{n}$. The *isodual isoeuclidean isospace* can be written for the case of diagonal isodual isounits

$$\hat{E}^d(\hat{r}^d,\delta^d,\hat{R}^d): \quad \hat{r}^d = \{ \hat{r}^k \times \hat{1}^d \} \equiv \{ -\hat{r}^k \}, \quad \hat{r}_k{}^d = \delta^d{}_{ki} \times^d \hat{r}^{di} = -\hat{r}_k ,$$

$$\hat{\delta}^d = \hat{T}^d \times \delta = -\hat{\delta}, \quad \delta = \text{diag.}\,(1, 1, 1), \quad \delta^d = \delta^{d\dagger}, \quad \hat{1}^d = \hat{T}^{d-1} = -\hat{1},$$

$$\hat{T}^d = \hat{T}^d(t, r, \dot{r}, \ddot{r}, ...) = \text{diag.}\,(-b_1{}^2, -b_2{}^2, -b_3{}^2) = -\hat{T} < 0, \quad b_k{}^d < ,$$

$$\hat{r}^{d2d} = (r^{di}\,\delta^d{}_{ij}\,r^{dj}) \times \hat{1}^d = (-xb_1{}^2x - yb_2{}^2y - zb_3{}^2z) \times \hat{1}^d \in \hat{R}^d . \quad (2.2.26)$$

Note that the above isospace admits as a particular case the novel *isodual Euclidean space* which occurs for $\hat{1}^d = I^d = -I$. Note also the two sequential steps for the characterization of the isodual isoinvariant,

$$\hat{r}^{2d} = \hat{r}^d{}_k\,\hat{x}^d\,\hat{r}^{dk} = r_k \times \hat{1}^d \times \hat{T}^d \times r_k \times \hat{1}^d = (r_k \times r^k) \times \hat{1}^d =$$

$$(r^i \times \hat{\delta}^d{}_{ij}\, r^j) \times \hat{1}^d = (-r^i\, \overline{\delta}_{ij} \times r^j) \times \hat{1}^d , \quad (2.2.27)$$

thus reproducing isodual isoinvariant (5.2.22d).

The *isodual isodistance* between two isodual isopoints $\hat{P}_1{}^d(\hat{x}_1{}^d, \hat{y}_1{}^d, \hat{z}_1{}^d)$ and $\hat{P}_2{}^d(\hat{x}_2{}^d, \hat{y}_2{}^d, \hat{z}_2{}^d)$ is the *negative-definite* isodual isoscalar

$$\hat{D}^d{}_{12} = \hat{|}\,(\hat{r}_1 - \hat{r}_2)\,\hat{|}^d =$$

$$= [(x_1 - x_2)^2\, b_1{}^2 + (y_1 - y_2)^2\, b_2{}^2 + (z_1 - z_2)^2\, b_3{}^2]^{1/2} \times \hat{1}^d = -\hat{D}_{12} \in \hat{R}^d . \quad (2.2.28)$$

Proposition 2.3: *The maps from Euclidean* $E(r,\delta,R)$ *to the isodual Euclidean space* $E^d(r^d,\delta^d,R^d)$ *and from the isoeuclidean* $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ *to the isodual isoeuclidean space* $\hat{E}^d(\hat{r}^d,\hat{\delta}^d,\hat{R}^d)$ *are antiautomorphic and imply the change of the sign of all positive-definite quantities.*

The above properties confirm the possibility for isoduality to provide a representation of motion backward in time in a causal way. In fact, it is evident that motion forward in time referred to a positive unit is fully equivalent, although antiautomorphic, to motion backward in time referred to a negative unit. Under these premises, any criticism on the isodual motion backward in time applies also to the motion forward in time.

We have studied until now the isoeuclidean geometry of Class I and, separately, the isodual geometry of Class II. These geometries can indeed remain separate and disjoint in all known physical applications. In fact, the former is applied to the representation of matter while the latter represents antimatter, while no particle is known to be able to transform into its antiparticle and viceversa. The production of particle-antiparticles pairs from the scattering of particles requires the tensorial product of isogeometries of Classes I and II which is a separate problem herein ignored.

The scientific scene in theoretical biology is different inasmuch as quantitative representations of certain biological structures, such as sea shells at bifurcations, *to admit both forward and backward directions of time for the same structure.*

As we shall see in Sect. V, the latter occurrence can be best represented with an *isoeuclidean geometry of Class III,* hereon referred to as a geometry with a generalized unit which can be positive as well as negative, thus permitting a continuous transformations from evolution forward in time to that backward in time and viceversa.

It should be stressed that the most appropriate isogeometry for the latter purpose would be that of Class IV which admits null isounits. The latter isogeometry is however singular, thus requiring rather delicate topological

$$V_1 \odot V_2 = x_1 x_2 + y_1 y_2 + z_1 z_2 , \tag{2.2.34}$$

is now lifted into the expression called *isoscalar product*

$$\hat{V}_1 \hat{\odot} \hat{V}_2 = (x_1 b_1^2 x_2 + y_1 b_2^2 y_2 + z_1 b_3^2 z_2) \times \hat{1} \in \hat{R}(\hat{n},+,*) . \tag{2.2.35}$$

Note that the isoscalar product preserves the original axioms, i.e.,

$$\hat{V}_1 \hat{\odot} \hat{V}_2 = \hat{V}_2 \hat{\odot} \hat{V}_1 , \quad \hat{V}_1 \hat{\odot} (\hat{V}_2 + \hat{V}_3) = \hat{V}_1 \hat{\odot} \hat{V}_2 + \hat{V}_1 \hat{\odot} \hat{V}_3 . \tag{2.2.36}$$

Moreover, the *isonorm* on $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ is expressible in terms of the isoscalar product via the rule

$$\hat{\uparrow} \hat{V} \hat{\uparrow} = (\hat{V} \hat{\odot} \hat{V})^{\hat{\frac{1}{2}}} \times \hat{1} \in \hat{R}(\hat{n},+,\hat{\times}). \tag{2.2.37}$$

Thus, the isocosinus of the isoangle formed by two intersecting isovectors can be written as the isotopy of the conventional case

$$\text{isocos } \hat{\alpha} = \frac{\hat{V}_1 \hat{\odot} \hat{V}_2}{\hat{\uparrow} \hat{V}_1 \hat{\uparrow} \hat{\times} \hat{\uparrow} \hat{V}_2 \hat{\uparrow}} . \tag{2.2.38}$$

Also, one can introduce the *directional isocosinuses* of a vector

$$\text{isocos } \hat{\alpha} = \hat{V}_1 / \hat{\uparrow} \hat{V} \hat{\uparrow} , \ \text{isocos } \hat{\beta} = \hat{V}_2 / \hat{\uparrow} \hat{V} \hat{\uparrow} , \ \text{isocos } \hat{\gamma} = \hat{V}_3 / \hat{\uparrow} \hat{V} \hat{\uparrow} . \tag{2.2.39}$$

Then, we have again the correct lifting of the corresponding conventional identity

$$b_1^2 \, \text{isocos}^2 \hat{\alpha} + b_2^2 \, \text{isocos}^2 \hat{\beta} + b_3^2 \, \text{isocos}^2 \hat{\gamma} = 1 . \tag{2.2.40}$$

Similarly the vectorial product $V_1 \wedge V_2$ is lifted in the expression called *isovectorial product*

spaces. In this section we shall solely study geometrical aspects and defer the applications to the last section.

The *isosurfaces* on the three-dimensional isoeuclidean space (5.2.5) are given by a straighforward isotopic image of ordinary curves and, as such, are reducible to algebraic equations in the coordinates of order higher than the first.

While in ordinary Euclidean space we have a large number of different surfaces, in the isoeuclidean space we have the dominance of the following notion:

Definition 2.9 [15]: *The "isosphere" in the three-dimensional isoeuclidean space* $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ *with diagonal isounit is the isotopic image of the ordinary sphere with equation*

$$\hat{r}^{\hat{2}} = \hat{x}^{\hat{2}} + \hat{y}^{\hat{2}} + \hat{z}^{\hat{2}} = \hat{R}^{\hat{2}}, \tag{2.2.46}$$

i.e.,

$$[\, x\, b_1^{\,2}(t, r, \dot{r}, \ddot{r}, ...)\, x + y\, b_2^{\,2}(t, r, \dot{r}, \ddot{r}, ...)\, y + z\, b_3^{\,2}(t, r, \dot{r}, \ddot{r}, ...)\, z\,] \times \hat{1} =$$

$$= \hat{R}^{\hat{2}} = R^2 \times \hat{1} \in \hat{R}, \tag{2.2.47}$$

where R *is an ordinary scalar. The isosphere is said to be of Class I, II, III, IV or V depending on the corresponding class of the isounit. The "isodual isosphere" is the isosphere in the isodual isospace* $\hat{E}^d(\hat{r}^d,\hat{\delta}^d,\hat{R}^d)$ *with equation*

$$\hat{r}^{d\hat{2}d} = \hat{x}^{d\hat{2}d} + \hat{y}^{d\hat{2}d} + \hat{z}^{d\hat{2}d} = \hat{R}^{d\hat{2}d}, \tag{2.2.48}$$

i.e.,

$$[\, x^d\, b_1^{\,2d}(t, r, \dot{r}, \ddot{r}, ...)\, x^d + y^d\, b_2^{\,2d}(t, r, \dot{r}, \ddot{r}, ...)\, y^d + z^d\, b_3^{\,2d}(t, r, \dot{r}, \ddot{r}, ...)\, z^d\,] \times \hat{1} =$$

$$= [\, - x\, b_1^{\,2}(t, r, \dot{r}, \ddot{r}, ...)\, x - y\, b_2^{\,2}(t, r, \dot{r}, \ddot{r}, ...)\, y - z\, b_3^{\,2}(t, r, \dot{r}, \ddot{r}, ...)\, z\,] \times \hat{1}^d =$$

$$= \hat{R}^{d2d} = R^{2d} \times \hat{1}^d \in \hat{R}^d, \tag{2.2.49}$$

It is then easy to see that the perfect sphere in Euclidean space is mapped into a surface with perfect spheridicity in isospace. In fact, the semiaxes of the original sphere $S_k = +1$, $k = x, y, z$, are lifted under isotopy to the values $\hat{S}_k = b_k^2$, $k = x, y, z$, thus yielding an ellipsoid. Jointly, the unit of each deformed semiaxis is lifted by the *inverse* amount, thus restoring the perfect spheridicity in isospace, with the understanding that the *diameter of the original sphere is changed.*

The above occurrence is directly expressed by the basic invariant (5.2.4) of the isotopic lifting of the Euclidean space realized according to the rules

$$\delta = \text{diag.}\,(1, 1, 1) \quad \to \quad \hat{\delta} = \text{diag.}\,(b_1^2, b_2^2, b_3^2)\,,$$

$$I = \text{diag.}\,(1, 1, 1) \quad \to \quad \hat{1} = \text{diag.}\,(b_1^{-2}, b_2^{-2}, b_3^{-2})\,, \tag{2.2.54}$$

or, equivalently, by the basic invariant under isotopies:

$$\delta \times I \quad \to \quad \hat{\delta} \times \hat{1} = \hat{T} \times \delta \times \hat{1} \equiv \delta \times I\,. \tag{2.2.55}$$

We can now begin to understand the representation of a hadron as an isosphere in isospace. Recall that in contemporary particle physics hadrons are represented as *perfectly spherical and perfectly rigid objects*, evidently as necessary conditions not to violate a pillar of quantum mechanics, the rotational symmetry SO(3).

The representation of a hadron as an isosphere then includes the perfectly spherical and rigid cases as trivial subcases and permits the additional representation of all the infinitely possible signature–preserving deformations of the sphere in such a way to preserve the basic rotational symmetry, as studied in details in Vol. II. In different terms, the geometric representation of a hadron as an isosphere permits a single, unified, rotationally invariant characterization of all possible actual, nonspherical shapes of a hadrons and all their infinitely possible deformations due to collisions or external fields.

It is easy to prove the following:

argument presented in the text, the perfect spheridicity can be best proved via the use of the isospherical coordinates and isotrigonometric functions of App. A. For the simpler case case of the *isocircle* of radius one in the isoplane $(\hat{x}, \hat{y})$ with diagonal isotopic element we have the *isopolar coordinates*

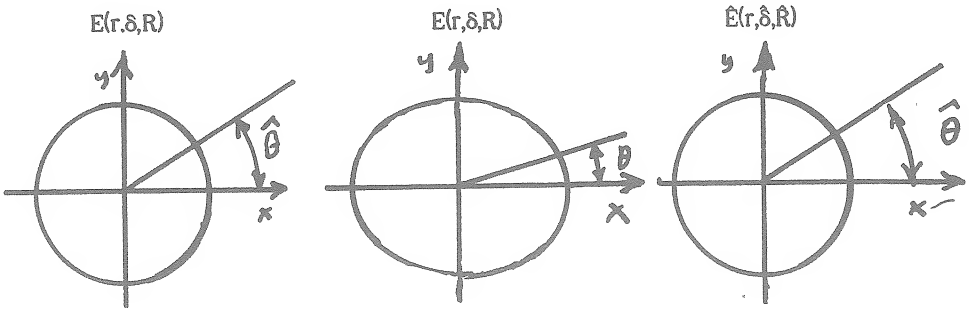
$$x = \text{isocos}\,\hat{\theta} = b_1^{-1}\cos\hat{\theta}, \quad y = \text{isosin}\,\hat{\theta} = b_2^{-1}\sin\hat{\theta}, \quad \hat{\theta} = b_1 b_2\,\theta, \quad (1)$$

where θ is the original angle of the circle prior to the isotopy. In this case the equation for the isocircle is reduced to the conventional form

$$\hat{r^2} = x\,b_1^2\,x + y\,b_2^2\,y = b_1^2\,\text{isocos}^2\hat{\theta} + b_2^2\,\text{isosin}^2\hat{\theta} =$$

$$= \cos^2\hat{\theta} + \sin^2\hat{\theta} = 1, \quad (2)$$

which can be schematically represented as follows



namely, the circle is first deformed into an ellipsoid in the original space and then reconstructed as a perfect circle in isospace. Note that the projection of the isocircle in the original space can also be represented with the coordinates $\bar{r}_k = r_k b_k$, $k = 1, 2$, on the conventional space $\bar{E}(\bar{r},\delta,R)$ with the self-evident identities

$$\hat{r^2} = x\,b_1^2\,x + y\,b_2^2\,y \equiv \bar{x}\,\bar{x} + \bar{y}\,\bar{y} = \bar{r}^2. \quad (3)$$

Similar results hold for the case of the isosphere via the use of the isospherical coordinates outlined in App. B, as well as for the isodual isosphere, as well as for the isosphere of Class III, the latter one requiring the use of the *isohyperbolic*

$$S\hat{O}(2.1): \quad x^1 b_1^2 x^1 - x^2 b_2^2 x^2 + x^3 b_3^2 x^3 = R^2,$$

5) All isodual sphere

$$SO^d(3): \quad -x^1 x^1 - x^2 x^2 - x^3 x^3 = -R^2,$$

6) All hyperbolic paraboloid

$$SO^d(2.1): \quad -x^1 x^1 + x^2 x^2 - x^3 x^3 = -R^2,$$

7) All isodual ellipsoids

$$S\hat{O}^d(3): \quad -x^1 b_1^2 x^1 - x^2 b_2^2 x^2 - x^3 b_3^2 x^3 = -R^2,$$

8) All isoduals deformations of the hyperbolic paraboloid

$$S\hat{O}^d(2.1): \quad -x^1 b_1^2 x^1 + x^2 b_2^2 x^2 - x^3 b_3^2 x^3 = -R^2, \tag{2.2.59}$$

The isosphere of Class IV unifies all the preceding surfaces plus

9) All possible cones in Euclidean space, *i.e.,*

$$SO(2.1): \quad x^1 x^1 - x^2 x^2 + x^3 x^3 = 0,$$

$$S\hat{O}(2.1): \quad x^1 b_1^2 x^1 - x^2 b_2^2 x^2 + x^3 b_3^2 x^3 = 0,$$

$$SO^d(2.1): \quad -x^1 x^1 + x^2 x^2 - x^3 x^3 = 0,$$

$$S\hat{O}^d(2.1): \quad -x^1 b_1^2 x^1 + x^2 b_2^2 x^2 - x^3 b_3^2 x^3 = 0. \tag{2.2.60}$$

hyperboloid (D), plus the related cones and isodual images here omitted for brevity. All these quadrics are unified into one, single, unique geometric notion in isospace.

It should be indicated that all *physical* applications known at this time are restricted to the isosphere of Class I, which unifies the sphere and all its ellipsoidical deformations and to the isodual isosphere of Class II, which unifies the isodual sphere and all its ellipsoidical deformations). This is due to the fact that there is no known physical event capable of altering, say, ellipsoids into hyperboloids, or viceversa.

Theorem 2.2 essentially states that *all quadrics (A)–(D) of Fig. 2.5 have the shape depicted only when expressed in the conventional Euclidean space, because when properly represented in isoeuclidean space they can all be reduced to perfect circles.*

This intriguing property should not be surprising for the reader now familiar with isotopic liftings. As it was the case for straight lines, *the isotopies of a sphere must remain a sphere as a necessary condition for the achievement of the isotopies themselves.* The unification of the sphere with all its infinitely possible ellipsoidical deformations then follows, with evidently broader unifications for higher classes.

One can now understand why distances which are very large in our perception of the universe in Euclidean space can become rather small in isospace. In fact, very large distances, say, in a hyperboloid are turned into relatively much shorter distances on the isosphere of Class III.

The reader should finally be aware that the unification of all quadrics into the isosphere is the geometric foundations of the unification of the compact SO(3) and non compact SO(2.1) symmetry into the isosymmetry $\hat{SO}(3)$ submitted by this author since the original proposal.

2.2.F: Connections with noneuclidean geometries

Recall that the isotopies lift the conventional Euclidean metric δ = diag. (1, 1, 1) into the isometrics $\hat{\delta} = \hat{T}\times\delta$ with a well behaved, but otherwise unrestricted functional dependence on time t, local coordinates r and their derivatives of arbitrary order, $\hat{\delta} = \hat{\delta}(t, r, \dot{r}, \ddot{r}, ...)$. The first noneuclidean property of the isoeuclidean geometry, apparently presented here for the first time, can be expressed as follows.

geometry and other *noneuclidean geometries* (see, e.g., ref.s [36,37] and quoted literature). As well known, Euclid's Fifth Axiom led to a historical controversy that lasted for two millennia, until solved by Lobacevski in a rather unpredictable way, via the introduction of a new, non–Euclidean geometry today appropriately called *Lobacevski geometry* (see [loc. cit.]).

As it is also well known, Lobacevski geometry is abased on certain liftings of Euclidean expressions, although defined on the conventional unit. Thus, the Lobacevski and isoeuclidean geometries are structurally different.

Nevertheless, it is important to understand that *the Lobacevski geometry is a particular case of the projection of the isoeuclidean geometry in the Euclidean plane.* To see this point consider the following celebrated transformations

$$x' = \frac{x + a}{1 + a\,x}, \qquad y' = \frac{y\,(1 - a^2)^{\frac{1}{2}}}{1 + ax}, \qquad |a| < 1, \tag{2.2.63}$$

which have the peculiar property of carrying straight lines into straight lines and circles into circles (see ref. [28] for details) *while keeping the unit the same.*

Now, the isoeuclidean space $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ of class I in two dimensions can be equivalently reinterpreted as an ordinary Euclidean plane $\bar{E}(\bar{r},\delta,R)$ in the new coordinates

$$\bar{x} = b_1(x, y, ...)\,x\,, \quad \bar{y} = b_2(x, y, ...)\,y\,, \tag{2.2.64}$$

under which we have the identity

$$\bar{x}\,\bar{x} + \bar{y}\,\bar{y} = x\,b_1^{\,2}\,x + y\,b_2^{\,2}\,y. \tag{2.2.65}$$

It is then evident that Lobacevski transformations (2.105) are contained as a particular case of the much larger class of isotransformations (2.107).

The connection between Lobacevski and isoeuclidean geometries can therefore be expressed by saying that:

A) *the Lobacevski geometry identifies "one" particular lifting of the*

possible maps of the Euclidean geometry, including singular maps for Class IV and discrete maps for Class V (universality), directly in the coordinates of the observer (direct universality).

In summary, the geometries studied in this section are the following:

1) Isoeuclidean geometry of Class I, used for the characterization of *biological structures evolving forward in time;*

4) Isodual Isoeuclidean geometry of Class II, used for the characterization of *biological structures evolving backward in time.*

5) Isogeometry of Class III, used for the characterization of *biological structures requiring time–inversions;*

6) Isoeuclidean geometries of Class IV, used for *biological structures with singularities;* and

7) Isoeuclidean geometry of Class V, used for the most general possible isotopic representation of biological structures.

2.3: Isotopies of classical and quantum methods

2.3.A: Introductory analytic profiles

We are now in a position to initiate the study of the third and final class of isotopic methods needed for biological applications, the isotopies of Newtonian, analytic and quantum mechanics.

The fundamental equations of contemporary mechanics are *Newton's equations* for a system of N particles with non–null masses in the second–order form

$$m_a \, dv_{ka} / dt = F^{SA}(t, r, v) + F^{NSA}{}_{ka}(t, r, v) , \qquad (2.3.1)$$

$$a = 1, 2, ..., N, \quad k = 1, 2, 3 \, (= x, y, z) , \quad v_{ka} = dr_{ka} / dt , \; m_a \neq 0 ,$$

where $\omega_{\mu\nu}$ is the familiar *exact canonical symplectic tensor* (see App. C)

$$(\omega_{\mu\nu}) = (\partial_\mu R^\circ_\nu - \partial_\nu R^\circ_\mu) = \begin{pmatrix} 0_{N\times N} & -I_{N\times N} \\ I_{N\times N} & 0_{N\times N} \end{pmatrix}. \qquad (2.3.8)$$

As it is also well known, the above methods, and the knowledge of the canonical action in particular, admit a unique and unambiguous map to operator formulations, resulting in contemporary quantum mechanical formulations.

The fundamental problem addressed in this section is that all the above classical and quantum methods are structurally insufficient for the representation of biological systems as outlined in Sect. 1 for numerous independent reasons, thus establishing the need to seek covering methods.

To begin, all contemporary classical and quantum formulations are solely applicable to local–differential systems, as demanded by the underlying topology and geometry, while we are primarily interested in non–local integral systems.

Also, the above methods are solely applicable to particles represented as perennial and immutable *massive points*, while we are interested in representing particles as extended, nonspherical and deformable.

Finally, the contemporary formulations of Lagrange's and Hamilton's equations can only represent a rather small class of Newton's equations (3.1) in the frame of the experimenter, those verifying the conditions of *variational self–adjointness* [4], while we are primary interested in representing well behaved, but otherwise arbitrary *nonpotential–nonselfadjoint systems.*

To the author's best knowledge, *all* attempts existing in the literature for the broadening of the above methods are equally insufficient for representations along Conditions 1)–5) of Sect. 1.2. These studies are reported in detail in monographs [4–5,7–8,9–10], and can only be outlined here.

The first broadening of the representational capabilities of the canonical action is given by the removal of the restriction that the representation occurs in the local coordinates (t, r, v) of the observer (called *direct representation*), thus permitting the use of the transformation theory and the representation of an equivalent system in another systems of coordinates (t',.r',v') (called *indirect representation*).

The first solution of the above problem was reached by the originators of analytic dynamics, Lagrange and Hamilton themselves, because they formulated their celebrated equations, not in the form of current use in the mathematics, physics and biology, Eq.s (3.6) and (3.7), but that with *external terms* representing precisely the forces F^{NSA}, as we shall reviewed in Part III.

According to this historical conception, the functions today called Lagrangian or Hamiltonian represent all potential forces, while all remaining contact–nonpotential forces are represented with the external terms. The above representation of Newton's equations does indeed verify the crucial requirement of occurring in the fixed inertial frame of the experimenter; the direct representation is manifestly universal; and its construction is truly simple and immediate. Unfortunately, analytic equations with external terms are not generally derivable from a conventional variational principle, thus losing the contemporary analytic character. More seriously, analytic equations with external terms imply the necessary abandonment of Lie's theory and related symplectic geometry with consequential major structural departures from contemporary methods, as indicated in Part III.

The first partial solution of the fundamental analytic problem here considered which preserves the Lie character of the underlying algebra, with consequential preservation of the symplectic geometry, was reached by Santilli in monograph [5] via a step–by–step generalization of Hamiltonian mechanics called, for certain historical reasons, *Birkhoffian mechanics.*

The main idea is to lift the canonical action principle (3.5) into the most general possible first–order action of the Pfaffian type

$$\delta A = \int_{t_1}^{t_2} [\, R_\mu(b)\, db^\mu - H(t, b)\, dt \,] = 0\,, \quad R = \{ P_k(x, p), Q^k(x, p) \}\,, \qquad (2.3.9)$$

which characterizes *Birkhoff's equations*

$$\left\{ \Omega_{\mu\nu}(b)\, \frac{db^\nu}{dt} - \frac{\partial H(t, b)}{\partial b^\mu} \right\} (P^\circ) = 0\,, \qquad (2.3.10)$$

Bloch'intsev, Fermi and others that the strong interactions have a nonlocal–integral component due to mutual overlapping of the wavepackets and charge distributions of hadrons. In fact, all hadrons have approximately the same size which coincides with the range of the strong interactions, $\approx$ 1 fm = 10^{-13} cm, thus requiring the necessary condition of mutual penetration of hyperdense particles, resulting in the most general known nonlinear integro–differential and nonhamiltonian equations.

In this memoir we review a solution of the fundamental problem herein considered along the latter lines, which was submitted for the first time in ref.s [32–35], and which is permitted by the isotopies of the differential calculus of Sect. II.1.

The most fundamental step is the first structural generalization of *Newton's* equations in *Newtonian* mechanics on known scientific record since their inception in the 1600's[7]. Such a generalization is then at the foundation of the generalized analytic and quantum methods. The limitations of local–differential and point–like character are inherent in the very structure of Newton's equations themselves, thus requiring a resolution at the Newtonian level and prior to any subsequent analytic or quantum profile.

In fact, classical Hamiltonian mechanics has been constructed to represent the conventional Newton's equations and, in turn, quantum mechanics has been constructed as an operator image of Hamiltonian mechanics. But the applicability of these mechanics is essentially restricted to local–differential and potential systems, while the advancement of knowledge in various disciplines is requesting the treatment of nonlocal–integral and nonpotential systems. It then follows that a possible broadening of contemporary dynamics must originate from its foundations, Newton's equations, which is done in the next section.

Our analysis is strictly local, owing to the need to identify methods which are specifically applicable in the given, fixed, inertial frame of the observer. The mathematical reformulation of the result in the coordinate–free language is left to interested mathematicians.

[7] All other generalizations of Newton's equations are of relativistic, gravitational, continuous and other types, but none of them is of *Newtonian* character, that is, conceived specifically for Newtonian mechanics.

$$\left\{ m\frac{dv_k}{dt} - \frac{d}{dt}\frac{\partial U(t,x,v)}{\partial v^k} + \frac{\partial U(t,x,v)}{\partial x^k} - F^{NSA}{}_k(t,x,v) \right\}^{NSA} =$$

$$= \left\{ m\frac{dv_k}{dt} - \frac{\partial U_k(t,x)}{\partial x^s}\frac{dv^s}{dt} + \frac{\partial U_o(t,x)}{\partial x^k} - F^{NSA}{}_k(t,x,v) \right\}^{NSA} = 0 , \tag{2.3.14}$$

namely, they are not in general derivable from Lagrange's or Hamilton's equations in the local chart {t, r, v}, as well known [4,5]. The extension to systems of n particles with masses m_k ($\neq 0$) is straighforward and will be ignored for brevity.

The first step in the application of all isotopies is the identification of the independent variables and related basic generalized units. The independent variables in Newton's equations are time t, coordinates r and velocities v. The fundamental assumption for the isotopies of Newton's equations is therefore the lifting of the conventional unit +1 for all the above variables into the following generalized units

$$\textit{Time isounit}: \ \hat{1}_t = \hat{T}_t^{-1} ,$$

$$\textit{Space isounit}: \ \hat{1}_r = \hat{T}_r^{-1} ,$$

$$\textit{velocity isounit}: \ \hat{1}_v = \hat{T}_v^{-1} . \tag{2.3.15}$$

For simplicity we assume hereon $\hat{1}_r \equiv \hat{1}_v = \hat{1}$, and therefore restrict the description to that characterized by the isounit of space and time only.

The representation space of the desired isotopic image of Newton's equations is given by the Kronecker product of isospaces

$$\hat{S}(t,\hat{r},\hat{v}) = \hat{E}(t)\times\hat{E}(\hat{r},\hat{\delta},\hat{R})\times\hat{E}(\hat{v},\hat{\delta},\hat{R}) , \tag{2.3.16}$$

with total isounit

nonlinear, nonlocal-integral and variationally non-self-adjoint Newton's equations (3.14) *always admit in a star-shaped neighborhood* $\hat{D}(\hat{S})$ *of a point* (t, r, v) *the representation in terms of the isotopic Newton's equations* (2.3.19)

$$\hat{m}\,\frac{\hat{d}\hat{v}_k}{\hat{d}\hat{t}} - \frac{\hat{d}}{\hat{d}\hat{t}}\,\frac{\hat{\partial}\hat{U}(\hat{t},\hat{r},\hat{v})}{\hat{\partial}\,\hat{v}^k} + \frac{\hat{\partial}\hat{U}(\hat{t},\hat{r},\hat{v})}{\hat{\partial}\,\hat{r}^k} = \tag{2.3.20}$$

$$= \hat{T}_k{}^i \left\{ m\,\frac{dv_i}{dt} - \frac{\partial U_i(t,r)}{\partial r^S}\,\frac{dr^S}{dt} + \frac{\partial U_o(t,r)}{\partial r^i} - F^{NSA}{}_i(t,r,v) \right\} =$$

Proof. When projected in the original space S(t, r, v), Eq.s (3.19) can be written

$$\hat{m}\hat{T}_t\,\frac{d\,(\hat{T}_k{}^i\,v_i)}{dt} - \hat{T}_t\frac{d}{dt}\,\hat{T}_k{}^i\,\frac{\partial\hat{U}(\hat{t},\hat{r},\hat{v})}{\partial\,\hat{v}^i} + \hat{T}_k{}^i\,\frac{\partial\hat{U}(\hat{t},\hat{r},\hat{v})}{\partial\,x^i} =$$

$$= \hat{m}\,\hat{T}_t\,\hat{T}_k{}^i\,\frac{d\,v_i}{dt} - \hat{T}_t\,\hat{T}_k{}^i\frac{\partial\hat{U}_i(t,r)}{\partial r^S}\,v^S + \hat{T}_k{}^i\frac{\partial\hat{U}_o(t,r)}{\partial\,r^i} + \hat{m}\,\hat{T}_t\,\frac{d\hat{T}_k{}^i}{dt}\,v_i = 0. \tag{2.3.21}$$

Sufficient conditions for identities (3.21) are then then given by

$$\hat{m}\,T_t\,dv_i\,/\,dt = m\,dv_i\,/\,dt\,,$$

$$\hat{T}_t\,\frac{\partial\hat{U}_i(t,x)}{\partial\,x^S}\,v^S = \frac{\partial U_i(t,x)}{\partial\,x^S}\,v^S\,,$$

$$\frac{\partial\hat{U}_o(t,x)}{\partial\,x^i} = \frac{\partial U_o(t,x)}{\partial\,x^i}\,,$$

$$\hat{m}\,\hat{T}_t\,\frac{d\hat{T}_k{}^i(t,x,...)}{dt}\,v_i = -\,\hat{T}_k{}^i\,F^{NSA}{}_i(t,x,v)\,.$$

which, under the assumed continuity and regularity conditions (see [5] for details)

symmetry is known to be a symmetry of *rigid bodies*. Form factors cannot therefore represent the *deformations* of perfectly spherical particles under sufficiently intense external interactions which is studied via other rather complicated procedures.

A first motivation for the studies presented in this memoir is to introduce a representation of actual *extended, nonspherical and deformable* shapes of particles at the primitive *Newtonian* level, which then persists under *classical* analytic representations as well as under maps to *first* quantization.

The isotopic Newton equations do indeed achieve these objectives by setting the foundations for possible new advances in classical and quantum treatment of biological structures. The objective is achieved via the new degrees of freedom of the generalized unit of the theory which are evidently absent in the conventional Newtonian, classical and quantum formulations.

As a simple case, suppose that the body considered is a rigid spheroidal ellipsoid with semiaxes n_1^2, n_2^2, n_3^2 = constants. Such a shape is directly represented by the isotopic element of the theory in the simple diagonal form (see Sec. II.2, i.e.,

$$\hat{T} = \text{diag.}\,(n_1^{-2}, n_2^{-2}, n_3^{-2}), \quad n_k = \text{const} > 0, \quad k = 1, 2, 3, \quad \hat{T}_0^{\,0} = 1. \qquad (2.3.25)$$

The representation of the shape in isospace $\hat{S}(\hat{t}, \hat{r}, \hat{v})$ is then embedded in the *isoderivatives* of the isotopic Newton equations and, when projected in the conventional space S(t, r, v) can be written

$$m\,\hat{T}_k^{\,i}\,\frac{d\,v_i}{dt} - \hat{T}_k^{\,i}\,\frac{d}{dt}\,\frac{\partial U_i(t, r)}{\partial\, v^i}\,v^s + \hat{T}_k^{\,i}\,\frac{\partial U_0(t, r)}{\partial\, r^i} = 0\,, \qquad (2.3.26)$$

namely, the shape terms $\hat{T}_k^{\,i}$ are admitted as factors.

Note that *the representation of shape occurs only in isospace because, when projected in the conventional Euclidean space, the shape terms cancel out by recovering the conventional point-like character of Newton's equations.* This illustrates the mechanism of the isotopy in the representation of shape.

always exist in the given coordinates (t, r, v) under the same conditions.

The following examples illustrate isorepresentation (3.21). The equation of the linearly damped particle in one dimension

$$m\, dv/dt + \gamma v = 0, \quad \gamma \in R(n,+,\times), \quad \gamma > 0, \tag{2.3.29}$$

admits isorepresentation (3.21) with values

$$\hat{T} = \hat{S}_0 e^{\gamma t/2m}, \quad \hat{T}_t = 1, \qquad U_k = U_0 = 0, \tag{2.3.30}$$

where $\hat{S}_0$ is a *shape factor,* e.g., of the spheroidal type (3.25) which is prolate in the direction of motion. In this way, the isotopic Newton equations represent: 1) the nonselfadjoint force $F^{NSA} = -\gamma v$ experienced by an object moving within a physical medium; 2) its extended character (which is necessary for the existence of the resistive force); and 3) the deformation of the original shape (in the case considered a perfect sphere) caused by the medium.

The equation for the linearly damped harmonic oscillator in one dimension in the x–axis

$$m\ddot{x} + \gamma \dot{x} + k x = 0, \quad k \in R(n,+,\times), \quad k > 0, \tag{2.3.31}$$

admits isorepresentation (3.21) with the values

$$\hat{T} = \hat{S}_0\, e^{\gamma t/2m}, \quad U_0 = -\tfrac{1}{2} k x^2, U_k = 0, \quad \hat{T}_t = 1, \tag{2.3.32}$$

where $\hat{S}_0$ represents the shape of the body oscillating within a resistive medium. The interested reader can construct a virtually endless variety of isorepresentations of non–self–adjoint forces. A systematic study will be conducted elsewhere.

Corollary 2.3.C: *The isotopic Newton equations permit the representation of nonlocal–integral forces when completely embedded in*

studies. However, the latter are based on a negative-definite norm (Sect. II.1), thus implying all negative-definite characteristics, beginning with *time moving backward*, but also including *negative-definite mass*, as well as all other characteristics. Finally, the reader should remember that the map from one to the other can be done via the isodual map

$$\hat{1}_t \to \hat{1}_t^d = -\hat{1}_t, \quad \hat{1} \to \hat{1}^d = -\hat{1}. \tag{2.3.35}$$

The study of the remaining aspects of isodual Newton's equations is left to the interested reader.

The *isodual Newton's equations* can then be written

$$\hat{m}^d \frac{\hat{d}^d \hat{v}_k^{\,d}}{\hat{d}t^d} - \frac{\hat{d}^d}{\hat{d}t^d} \frac{\partial^d \hat{U}^d(t^d, \hat{r}^d, \hat{v}^d)}{\hat{\partial}\, \hat{v}^{kd}} + \frac{\partial^d \hat{U}^d(t^d, \hat{r}^d, \hat{v}^d)}{\hat{\partial}\, \hat{r}^{kd}} = \tag{2.3.36}$$

$$= -\hat{T}ki \left\{ m \frac{dv_i}{dt} - \frac{\partial U_i(t, r)}{\partial r^S} \frac{dr^S}{dt} + \frac{\partial U_o(t, r)}{\partial r^i} - FNSAi(t, r, v) \right\} = 0$$

where all operations are isodual, including products and division.

2.3.C: Isotopies of Lagrangian methods

We now show the derivability of the isotopic Newton equations from a first-order isovariational principle and then study the isotopies of Lagrange's and Hamilton's mechanics.

Recall that the canonical action functional (3.5) is called of *first-order* when its integrand depends on the first-order derivatives (velocities, L = L(t, r, v); it is called of *second-order* when the integrand depends on the second-order derivative (the accelerations a), L = L(t, r, v, a); and so on.

Proposition 2.6: *All action functionals of second or higher order in Euclidean space* $E(t)\times E(r,\delta,R)\times E(v,\delta,R)$ *whose integrand is sufficiently*

where the explicit expression of the isoderivatives is understood.

The isotopies of the remaining aspects of the calculus of variations (see, e.g., Bliss [42]) with consequential isotopies of the optimal control theory are intriguing and significant for theoretical biology, but they cannot be studied here for brevity.

We shall say that the isotopic Newton equations (3.19) admit a *direct isoanalytic representation* in terms of the isolagrange equations when there exists one isolagrangian $\hat{L}(t, \hat{r}, \hat{v})$ under which all the following identities occur

$$\left\{ \frac{\hat{d}}{\hat{d}t} \frac{\hat{\partial} \hat{L}(t,\hat{r},\hat{v})}{\hat{\partial} \hat{v}^k} - \frac{\hat{\partial} \hat{L}(t, \hat{r}, \hat{v})}{\hat{\partial} \hat{r}^k} \right\}^{IS} =$$

$$= \left\{ \hat{m} \frac{\hat{d}v_k}{\hat{d}t} - \frac{\hat{\partial} \hat{U}_k(t, \hat{r})}{\hat{\partial} \hat{x}^i} \frac{\hat{d} \hat{r}^i}{\hat{d}t} + \frac{\hat{\partial}\hat{U}_o(t, \hat{r})}{\hat{\partial} \hat{x}^k} \right\}^{ISOSA} =$$

$$= \hat{T}_k{}^i \left\{ m \frac{dv_i}{dt} - \frac{\partial U_i(t, r)}{\partial r^S} \frac{dr^S}{dt} + \frac{\partial U_o(t, r)}{\partial r^i} - F^{NSA}{}_i(t, r, v) \right\} = 0$$

$$\hat{L}(t, \hat{r}, \hat{v}) = \tfrac{1}{2} \hat{m} \hat{v}^k \hat{v}_k - \hat{U}, \quad \hat{U} = \hat{U}_k(t, \hat{r}) \hat{v}^k + \hat{U}_o(t, \hat{r}), \tag{2.3.40}$$

Theorem 2.4 (Direct Universality of Isolagrangian Mechanics). *All possible sufficiently smooth and regular, nonlinear, nonlocal and non-first-order Lagrangian systems always admit a direct isorepresentation in isospace via the isolagrange equations (3.39) in a star-shaped neighborhood of a point of their variables.*

Proof. The universality of the isorepresentation follows from the fact that conditions (3.22) always admit solution in the unknown functions, as well as from the universality of the isoaction. **q.e.d.**

Recall that *Newtonian systems* are usually referred to systems with local-differential forces depending at most on velocities. Theorem II.3.2 includes also

$$\hat{1}_2 = \text{diag.}\,(\hat{1}_t, \hat{1}, \hat{T}), \quad \hat{T}_2 = \text{diag.}\,(\hat{T}_t, \hat{T}, \hat{1}). \tag{2.3.43}$$

where the use of the isotopic element $\hat{T}$, in liew of the isounit $\hat{1}$, for the linear momentum is requested by its contravariant character.

It should be indicated that, in view of the independence of the variables $\hat{p}_k$ from $\hat{r}^k$, we can introduce a new isounit $\hat{W} = \hat{Z}^{-1}$ for the isospace $\hat{E}(\hat{p},\hat{\delta},\hat{R})$ which is different than the unit $\hat{1} = \hat{T}^{-1}$ of isospace $\hat{E}(\hat{r},\hat{\delta},\hat{R})$, in which case the total unit is $\hat{1}_2 = \text{diag.}\,(\hat{1}_t, \hat{1}, \hat{W})$. Selection (4.43) is the simplest possible case with $\hat{W} \equiv \hat{T}$ which is recommendable on geometric grounds [35]. Other alternatives belong the the problem of the degrees of freedom of the isotopic theories which is not studied at this time for brevity.

We now introduce the *isocanonical momentum* via the following prescriptions

$$\hat{p}_k = \frac{\hat{\partial}\,\hat{L}(\hat{t}, \hat{r}, \hat{v})}{\hat{\partial}\,\hat{v}^k} = \hat{m}\,\hat{v}_k - \hat{U}_k(\hat{t}, \hat{r}), \tag{2.3.44}$$

under the condition of being regular, thus invertible, in a (2N+1)–dimensional region $\hat{D}$ of points $(\hat{t}, \hat{r}, \hat{p})$

$$\text{Det.}\left(\frac{\hat{\partial}^2\hat{L}(\hat{t}, \hat{r}, \hat{v})}{\hat{\partial}\,\hat{v}^i\,\hat{\partial}\,\hat{v}^j} \right) (\hat{\Re}) \neq 0. \tag{2.3.45}$$

thus admitting a unique set of implicit functions $\hat{v}^k = \hat{f}^k(\hat{t}, \hat{r}, \hat{p})$. The *isolegendre transform* of Class III can then be defined by [32,34]

$$\hat{L}[\hat{t}, \hat{r}, \hat{v}(\hat{t}, \hat{r}, \hat{p}) = \hat{p}_k\,\hat{v}^k(\hat{t}, \hat{r}, \hat{p}) - \tfrac{1}{2}\,\hat{m}\,\hat{v}_i(\hat{t}, \hat{r}, \hat{p})\,\hat{v}^j(\hat{t}, \hat{r}, \hat{p}) +$$

$$+ \hat{U}_k(\hat{t}, \hat{r})\,\hat{v}^k(\hat{t}, \hat{r}, \hat{p}) + \hat{U}_o(\hat{t}, \hat{r}) = \hat{p}_k\,\hat{p}^k / 2\hat{m} + \hat{V}^k(\hat{t}, \hat{r})\,\hat{p}_k + \hat{V}^o(\hat{t}, \hat{r}) = \hat{H}(\hat{t}, \hat{r}, \hat{p}) \tag{2.3.46}$$

We are now equipped to study the *isotopies of Hamilton's principle*. By using the unified variables $\hat{b} = \{\hat{b}^\mu\} = \{\hat{r}^k, \hat{p}_k\}$, $\hat{c}^\mu = \hat{d}\,\hat{b}^\mu / \hat{d}t$, and by

The above equations can be simply written in the following covariant and contravariant forms, respectively,

$$\omega_{\mu\nu} \frac{\hat{d}\,\hat{b}^{\nu}}{\hat{d}t} = \frac{\hat{\partial}\,\hat{H}(t,\hat{b})}{\hat{\partial}\hat{b}^{\mu}},$$

$$\frac{\hat{d}\,\hat{b}^{\mu}}{\hat{d}t} = \omega^{\mu\nu} \frac{\hat{\partial}\,\hat{H}(t,\hat{b})}{\hat{\partial}\hat{b}^{\nu}}, \qquad (2.3.51)$$

where the quantities

$$(\omega_{\mu\nu}) = \left(\frac{\hat{\partial}\hat{R}^{\circ}_{\nu}}{\hat{\partial}\hat{b}^{\mu}} - \frac{\hat{\partial}\hat{R}^{\circ}_{\mu}}{\hat{\partial}\hat{b}^{\nu}} \right) = \begin{pmatrix} 0_{N\times N} & -I_{N\times N} \\ I_{N\times N} & 0_{N\times N} \end{pmatrix},$$

$$(\omega^{\alpha\beta}) = \left(\frac{\hat{\partial}\hat{R}^{\circ}_{\nu}}{\hat{\partial}\hat{b}^{\mu}} - \frac{\hat{\partial}\hat{R}^{\circ}_{\mu}}{\hat{\partial}\hat{b}^{\nu}} \right)^{-1} = \begin{pmatrix} 0_{N\times N} & I_{N\times N} \\ -I_{N\times N} & 0_{N\times N} \end{pmatrix}, \qquad (2.3.52)$$

are the *conventional* covariant and contravariant canonical tensors, respectively, which hold in view of the properties of the isodifferential calculus

$$\hat{\partial}\hat{R}^{\circ}_{\nu} / \hat{\partial}\hat{b}^{\mu} \equiv \partial R^{\circ}_{\nu} / \partial b^{\mu}. \qquad (1.3.53)$$

The proof of the following property via the preceding results is then evident.

Theorem 2.5 (Direct Universality of the Isohamilton Equations): *Under sufficient smoothness and regularity conditions, all nonlinear, nonlocal and nonhamiltonian first-order systems admit a direct representation in terms of the isohamilton equations in isospace in the local coordinates of the experimenter.*

The equivalence of the isolagrangian and isohamiltonian equations under the assumed regularity and invertibility of the isolegendre transform can be proved as in the conventional case (see, e.g., [4], Sect. 3.8).

with elements $\hat{\Upsilon}_\mu{}^\nu$

$$\Omega^{\mu\nu} \equiv \omega^{\alpha\beta}\, \hat{\Upsilon}_\beta{}^\nu , \qquad (2.3.57)$$

under which Birkhoff's equations coincides with the isohamilton's equations (3.51b) for $\hat{1}_t = 1$. As a result, *Birkhoffian mechanics is a particular case of the isohamiltonian mechanics.*

Despite these similarities, it should be indicated that the isohamiltonian mechanics is considerably broader than the Birkhoffian mechanics. In fact, the former is based on an action of arbitrary order, while the latter necessarily requires a first-order action. Also, the former can represent integral forces, while the latter cannot (because the underlying geometry, the symplectic geometry in its most general possible exact realization) only admits local-differential systems. Finally, the former is based on a broader mathematics, the isodifferential calculus, while the latter is based on conventional mathematics.

Note that the isotopic Hamilton-Jacobi equations (3.55) imply the properties

$$\hat{\partial}\, \hat{A}^\circ / \hat{\partial}\, \hat{p}_k \equiv 0 , \quad k = 1, 2, ..., N , \qquad (2.3.58)$$

which are *necessary* for a correct isotopy of quantization studied in the next section.

By comparison, the Pfaffian principle implies the following *Birkhoffian Hamilton-Jacobi equations* (studied in detail in [15])

$$\frac{\partial A}{\partial t} + H(t, r, p) = 0 , \qquad \frac{\partial A}{\partial r^k} - P_k(xr\, p) , \qquad \frac{\partial A}{\partial p_k} - Q^k(r, p) = 0 \qquad (2.3.59)$$

for which $\partial A/\partial p_k \neq 0$ and the operator image of the theory would imply "wavefunctions" depending on both r and p indicated in Sect. II.3.1. As a result, Birkhoffian mechanics is not a suitable classical foundation for the isotopies of quantum mechanics, and the isohamiltonian mechanics is much preferable.

We should briefly mention that the isohamiltonian mechanics provides a

simplest possible isotopies for which the isounits of the independent variables $\hat{p}_k$ and $\hat{r}^k$ are inverse of each other. The use of different isounits for $\hat{p}_k$ and $\hat{r}^k$ evidently implies further differences between the isotopic and conventional brackets.

We should also indicate that the isohamiltonian mechanics provides a classical realization of the *Lie-Santilli isogroups or isosymmetries* [11-14,21]. In fact, the integrated form of Eq. (3.51b) yields the time evolution of a quantity $\hat{A}(\hat{t})$ in isospace

$$\hat{A}(t) = \exp\left\{ t \left[\frac{\partial \hat{H}}{\partial \hat{r}} \hat{T} \frac{\partial}{\partial \hat{p}} - \frac{\partial \hat{H}}{\partial \hat{r}} \hat{T} \frac{\partial}{\partial \hat{p}} \right] \right\} \hat{A}(0) , \tag{2.3.64}$$

which constitutes a one-parameter Lie-Santilli isogroup, with similar structures in higher dimensions.

Again, we have presented the isohamilton equations fin their Class II version including those of Classes I and II. The latter can be explicit written in terms of the *isodual isohamilton equations*

$$\omega_{\mu\nu}{}^{d} \frac{\hat{d}^d \hat{b}^{\nu\, d}}{\hat{d}t} - \frac{\partial^d \hat{H}^d(\hat{t}^d, \hat{b}^d)}{\partial \hat{b}^\mu} =$$

$$= - \omega_{\mu\nu} \frac{\hat{d}\, \hat{b}^\nu}{\hat{d}t} - \frac{\hat{\partial}\, \hat{H}(\hat{t}, \hat{b})}{\partial \hat{b}^\mu} , \tag{2.3.65}$$

and their provide a direct representation of the isodual newton's equations of Sect. II.3.2.

2.3.E. Isoquantization.

Since theoretical biology is today studied at both classical and quantum levels [1-3], it is important for completeness of this memoir to outline the *isotopies of quantum mechanics* also known under the name of *hadronic*

$$i\,\hat{\partial}\,\hat{\psi}\,/\,\hat{\partial}\hat{t} \;=\; i\,\hat{T}_t\,\partial\hat{\psi}\,/\,\partial t \;=\; \hat{H}\hat{T}\,\hat{\psi} \;=\; \hat{H}\,\hat{\times}\,\hat{\psi}\,, \qquad (2.3.69)$$

the *isoschrödinger equation in the momentum*

$$\hat{p}_k\,\hat{T}\,\hat{\psi} \;=\; \hat{p}_k\,\hat{\times}\,\hat{\psi} \;=\; -\,i\,\hat{\partial}\,\hat{\psi}\,/\,\hat{\partial}\hat{x}^k \;=\; -\,i\,\hat{T}_k{}^i\,\partial\hat{\psi}\,/\,\partial r^i\,, \qquad (2.3.70)$$

and the related *fundamental isocommutation rules*

$$[\,\hat{b}_\mu\,\hat{,}\,\hat{b}_\nu\,] \;=\; \hat{b}_\mu\,\hat{T}\,\hat{b}_\nu \;-\; \hat{b}_\nu\,\hat{T}\,\hat{b}_\mu \;=\; \omega_{\mu\nu}\,\hat{1}\,, \qquad \hat{b} \;=\; \{\,\hat{x}_k,\,\hat{p}_k\,\} \qquad (2.3.71)$$

Note the abstract identity between the conventional and the isotopic quantization, as illustrated by the preservation of the *conventional* symplectic structure $\omega_{\mu\nu}$ in the transition from the conventional commutation rules (3.67) to the isotopic ones (3.71).

The reader should be aware that rigorous studies conducted via the isotopies of symplectic quantization by a number of authors have confirmed the uniqueness of main results of this section.

2.3.F: Hadronic mechanics

The covering of quantum mechanics which emerges from the isoquantization of isohamiltonian mechanics, called hadronic mechanics, is based on the following fundamental structures defined for isounits of Class III [10]:

1) Enveloping operator algebra $\hat{\xi}$ with generic elements $\hat{A}$, $\hat{B}$, ... (which are the same polynomials in r and p of the quantum algebra only written in isospace) called *isoassociative envelope* and characterized by the isoassociative product $\hat{A}\hat{\times}\hat{B} = \hat{A}\times\hat{T}\times\hat{B}$ with isounit $\hat{1} = \hat{T}^{-1}$;

2) The isofields $\hat{C}(\hat{c},+,\hat{\times})$ of isocomplex numbers $\hat{c} = c\times\hat{1}$, or its isoreal particularization $\hat{R}(\hat{n},+,\hat{\times})$; and

3) The *isohilbert space* $\hat{\mathcal{H}}$ with *isostates* $\hat{\psi}$. $\hat{\phi}$, ..., and *isoinner product*

which $\hat{\xi} \equiv \xi$, $\hat{C} \equiv C$, $\hat{\mathcal{H}} \equiv \mathcal{H}$, etc. This guarantees the axiomatic consistency of hadronic mechanics to such an extent that *criticisms on the axiomatic structure of hadronic mechanics ultimately are criticisms on quantum mechanics.* Hadronic mechanics merely provides a more general nonlinear, nonlocal and nonpotential *realization* of the same quantum axioms.

All quantum notions, rules and properties are therefore preserved by hadronic mechanics. For instance, it has been shown that *all quantities which are Hermitean for quantum mechanics remain Hermitean for hadronic mechanics* [10]. Therefore, all conventional characteristics which are observables for quantum mechanics remain observable under isotopies. Even the explicit form of the operators representing physical quantities remain the same (such as coordinates, momentum, angular momentum, energy, etc.) and only the *operations* among them are generalized.

Similarly, hadronic mechanics preserves the notion of unitarity, only realized in the more general *isounitary law*

$$\hat{U} \hat{\times} \hat{U}^\dagger = \hat{U}^\dagger \hat{\times} \hat{U} = \hat{1} . \tag{2.3.77}$$

Note in particular that *any nonunitary transform can be identically rewritten in an isounitary form.* In fact, any given nonunitary transform $U \times U^\dagger \neq I$, can be identically rewritten

$$U \times U^\dagger = \hat{1} = \hat{T}^{-1} \neq I, \; U = \hat{U} \times \hat{T}^{1/2}, \; U \times U\dagger \equiv \hat{U} \times \hat{T} \times \hat{U}^\dagger = \hat{1} . \tag{2.3.78}$$

This illustrates that *nonunitarity is not an axiomatic property because it can be made to disappear under isotopies* [9].

The *isoexpectation values* of an observable $\hat{H}$ are given by

$$< \hat{H} > = < \hat{\psi} | \hat{\times} \hat{H} \hat{\times} | \hat{\psi} > / < \hat{\psi} | \hat{\times} | \hat{\psi} > = E \in R(n,+,\times) , \tag{2.3.79}$$

(where one should note the *necessity*, for consistency, that all products are isotopic) and they results to be *ordinary real numbers*[8] [10]. Moreover, the

[8] This is due to the fact that the isounits in the ratio cancel out, yielding an ordinary

reconstruction also holds for other properties, such as the reconstruction of exact space–time and internal symmetries when believed to be conventionally broken [10].

This illustrates the statement in Part I that the isotopies map linear, local and canonical structures into the most general possible nonlinear, nonlocal and noncanonical form and are capable of restoring linearity, locality and canonicity in generalized spaces over generalized fields.

We should also indicate that hadronic mechanics is invariant under its own *isounitary* transforms, as established by the invariance of the isounit,

$$\hat{1} \quad \to \quad \hat{1}' = \hat{U} \hat{\times} \hat{1} \hat{\times} \hat{U}^{\dagger} \equiv \hat{1} , \tag{2.3.84}$$

the invariance of the isoassociative product

$$\hat{A} \hat{\times} \hat{B} \to \hat{U} \hat{\times} (\hat{A} \hat{\times} \hat{B}) \hat{\times} \hat{U}^{\dagger} = \hat{A}' \hat{\times} \hat{B}', \quad \hat{A}' = \hat{U} \hat{\times} \hat{A} \hat{\times} \hat{U}^{\dagger} , \tag{2.3.85}$$

with consequential invariance of the isoschrödinger and isoheisenberg equations, isoexpectation values, isospecial functions, etc.

Most suggestive is the fact that *the isoexpectation values of the isounit of Class III reproduce the conventional number* +1

$$< \hat{1} > = < \hat{\psi} | \hat{\times} \hat{1} \hat{\times} | \hat{\psi} > / < \hat{\psi} | \hat{\times} | \hat{\psi} > =$$

$$= < \hat{\psi} | \times \hat{T} \times \hat{T}^{-1} \times \hat{T} | \hat{\psi} > / < \hat{\psi} | \hat{T} \times \hat{\psi} > \equiv + 1 \in R . \tag{2.3.86}$$

This indicates the "hidden" character of hadronic mechanics, which emerges as being a form of completion of quantum mechanics essentially along the celebrated argument by Einstein, Podolsky and Rosen. In particular, the isoeigenvalue equation (3.80) is an explicit and concrete realization of the "hidden variables".

The implications of property (3.86) are far reaching. By recalling from the preceding section that the isounit is a generalization of Planck's constant,

the synthesis of a neutron as occurring inside stars from the sole use of protons and electrons along Rutherford's historical legacy, which is notoriously impossible for quantum mechanics; and the interpretation of numerous other phenomena in nuclear physics, particle physics, statistical physics, astrophysics, gravitation, cosmology, superconductivity and other fields [10].

The inequivalence of quantum and hadronic mechanics is finally established by the fact that *the latter is a nonunitary image of the former,* as established by the fact, e.g., that a nonunitary image of canonical commutation rules yields exactly the isotopic rules with the correct Hermiticity of the isotopic elements and isounit,

$$U \times U^\dagger = \hat{1} = I^\dagger \neq I, \quad \hat{T} = (U \times U^\dagger)^{-1} = \hat{T}^\dagger = \hat{1}^{-1},$$

$$U \times (r \times p - p \times r) \times U^\dagger = \bar{r} \times \hat{T} \times \bar{p} - \bar{p} \times \hat{T} \times \bar{r} = i\hat{T}^{-1}, \quad \bar{r} = U \times r \times U^\dagger. \quad (2.3.89)$$

The inequivalence between quantum and hadronic mechanics is then established by the fact that *all quantum mechanical spectra of eigenvalues are altered by nonunitary transforms.*

The above property also establishes the nontriviality of hadronic mechanics while possessing considerable pragmatic value. In fact, any quantum mechanical model of biological systems can be easily lifted into the isotopic form precisely via the use of nonunitary transforms with a nonlinear and nonlocal structure. The emerging model is axiomatically consistent provided that the transforms are systematically applied to *all* aspects, including numbers, fields, states, vector spaces, inner product, etc., as outlined in this Part II.

In summary, the isotopic element $\hat{T}$ of the isonewton equations is preserved in its entirety at the level of the isohamiltonian representation, and then persists under isoquantization at the operator counterpart.

This implies that hadronic mechanics preserves all the representational capabilities of the primitive isonewton's equations, with particular reference to the representation of extended, nonspherical and deformable particles with linear and nonlinear, local and nonlocal and hamiltonian or nonhamiltonian interactions.

Another generalization is given by the so–called q–deformations of quantum mechanics, which are based on the deformation of the associative product

$$A \times B \to A \hat{\times} B = q \times A \times B, \tag{2.3. 92}$$

where q is a number, while the original unit is left unchanged. Even though mathematically impeccable, the above generalization is afflicted by drawbacks so serious to prevent any meaningful application known at this writing. In fact, the time evolution of q–deformations is generally nonunitary. It then follows that q–deformations: 1) do not possess an invariant unit I, thus being inapplicable to actual experiment; 2) do not possess observables because Hermiticity can be easily proved to be non–preserved in time; 3) do not possess invariant data elaborations because the q–number becomes an operator under the time evolution of the theory; etc.

Again, the q–deformations can be *identically* reformulated in an isotopic form, by merely lifting the unit into the form

$$\hat{1} = q^{-1}. \tag{2.3.93}$$

and then subjecting the entire theory to an isotopic reformulation as studied in this Sect. II, which resolves *all* above problematic aspects, as one can verify.

All other generalizations of quantum mechanics known to this author, that is, generalized theories which are *nonunitarily equivalent to quantum mechanics,* are afflicted by other equally serious problematic aspects in applications (see [10] for brevity).

The above problematic aspects of other generalizations are the reason why, after decades of study of the problem, this author selected the isotopies as the only known methods for resolving structural inconsistencies.

$$\hat{1}^{>} = (\hat{1}^{<})^{t}. \tag{3.4}$$

We should finally indicate that in specific applications the interconnecting map can be realized via other conjugation, such as the inverse,

$$\hat{1}^{>} = (\hat{1}^{<})^{-1}. \tag{3.5}$$

The isoassociative product $A\hat{\times}B = A\times\hat{1}\times B$ is then lifted into the corresponding two products

$$A > B = A \times S \times B, \quad A < B = A \times R \times B, \tag{3.6}$$

for which the quantities $\hat{1}^{>}$ and $\hat{1}^{<}$ are the corresponding left and right generalized units

$$\hat{1}^{>} > A = A > \hat{1}^{>} \equiv A, \quad A < \hat{1}^{<} = \hat{1}^{<} < A \equiv A, \tag{3.7}$$

in which case (only), the quantities $\hat{1}^{>}$ and $\hat{1}^{<}$ are called *forward and backward genounits,* and the related quantities S and R are called the *forward and backward genotopic elements, respectively.*

The emerging classical and quantum genotopic methods verify all conditions 1)–5) of Sect. I.2, provided that the genotopies are applied to each and every aspect, including: fields, angles, trigonometric and hyperbolic functions, differential calculus, vector and metric spaces, etc.

We should insist here that, "mixed methods" composed by the genotopic formulation of certain aspects and conventional formulations for others are afflicted by a number of inconsistencies which render them unusable for applications, and which usually remain undetected by non–expert ion the field.

An illustrative example is given by "mixed methods" given by a genometric space defined over conventional field. Such a mixture is structurally inconsistent because it implies a generalized *nonsymmetric* metric defined with respect to a *symmetric* unit which is afflicted by a host of geometrical problems, besides being noninvariant in time, thus having no practical value.

the isonumbers of Sect. II. The correct elaboration of the methods therefore requires their *entire* referral to genonumbers, although the final numbers to be experimentally verified result to be ordinary.

The genonumbers were first indicated by Santilli in memoir [6b] of 1978 and then studied in detail in memoir [15].

Let $F(\alpha,+,\times)$ be a conventional field with conventional sum $+$, multiplication $\alpha\beta = \alpha\times\beta$, additive unit 0 and multiplicative unit 1. In Sect. II we have reviewed a generalization of the multiplication into the isotopic form $\alpha\hat{\times}\beta = \alpha\times T\times\beta$. where both products $\alpha\times\beta$ and $\alpha\hat{\times}\beta$ are based on the assumption that they apply irrespective of whether α multiplies β from the left, or β multiplies α from the right. We can therefore introduce the following:

Definition 3.1: Ordering of the multiplication [15]: *The multiplication of two numbers* α *and* β *is ordered to the right, and denoted* $\alpha>\beta$, *when* α *multiplies* β *to the right, while it is ordered to the left, and denoted* $\alpha<\beta$ *when* β *multiplies* α *from the left.*

Note that the above ordering is compatible with other properties and axioms of the number theory. As an example, if the original field F is commutative, it remains commutative after the above ordering. In fact, if $\alpha\beta = \beta\alpha$, then $\alpha>\beta = \beta>\alpha$ and $\alpha<\beta = \beta<\alpha$. The same occurrence holds for other properties, such as associativity while the verification of the left and right distributive laws is evident. Thus, the definition of isonumbers can be reformulated under ordering by characterizing fully acceptable fields.

We can therefore restricts the definition of of ordinary field to that ordered to the left, $^{<}F(\alpha,+,<)$ or that ordered to the right $F^{>}(\alpha,+,>)$ and similarly restrict the isofields to those to the left $^{<}\hat{F}(\hat{\alpha},+,<)$ and those to the right $\hat{F}^{>}(\hat{\alpha},+,>)$. Each of these restricted formulations verifies all axioms for a field.

The point at the foundations of genotopic methods is that *the multiplications of the same numbers in different orderings can be different,* $\alpha>\beta \neq \beta<\alpha$, *while preserving all axioms of a field* [15]. This occurrence implies the existence of a dual generalization of fields and isofields, one for ordering to the

differentiation of these two multiplications; and C) the assumption of an interconnecting map representing time-reversal.

It is evident that the genotopic methods imply four different notions of time

Forward genotime $\hat{t}> = t\hat{1}_t>$, $\hat{1}_t> = S_t^{-1}$,

Conjugated forward genotime $<\hat{t} = t<\hat{1}_t$, $<\hat{1}_t = R_t^{-1} = (\hat{1}_t>)^\dagger$,

Backward genotime $\hat{t}>^d = -\hat{t}>$, $\hat{1}>^d = -\hat{1}>$,

Conjugated backward genotime $<\hat{t}^d = -<\hat{t}$, $<\hat{1}^d = -<\hat{1}$ (3.11)

where the quantities $<\hat{1}_t>$ and $<\hat{1}_t>$ are the *time genounits* with corresponding time products written in the unified form $<\hat{t}_1> <>_t <\hat{t}_2>$ and $<\hat{t}_1>^d <>^d_t <\hat{t}_2>^d = - <\hat{t}_1> <>_t <\hat{t}_2>$. Note that the space and time genounits are structurally different because the former are three-dimensional matrices while the latter are scalars.

The time genounits can therefore be ordinary complex functions, such as $f_1(r, ..) + i\, f_2(r, ...)$, where f_1 and f_2 are real. The four possible "time arrows" are then given by

$$\hat{t}> = t(f_1+if_2),\ <\hat{t} = t(f_1+if_2),\ \hat{t}>^d = t(-f_1+-if_2),\ ^d<\hat{t} = t(-f_1+if_2). \qquad (3.12)$$

Note that *the above four complex "time arrows" are represented by complex genounits, while time itself remains the ordinary real quantity* . As we shall see in the applications, rather than being a mere mathematical curiosity, the above characterizations of time raise intriguing and novel *experimental* issues.

The entire theory of isospaces of Sect. 2 admits a consistent and significant genotopic covering. Let $E(r,\delta,R)$ be a conventional Euclidean space and $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ its family of isotopes. Then, we can introduce the following *forward and backward genoeuclidean spaces of Class III*

Note the preservation of the original geometric axioms exactly as for the isotopies, despite the fact that the genometrics are no longer totally symmetric. This is permitted by the main mechanism of genotopies, the deformation of the originally symmetric metric via a nonsymmetric 3×3 matrix, $\delta \to \hat{\delta}^{>} = R \times \delta$, $R^{t} \neq R$, while the original unit is deformed of the inverse amount, $I = \text{diag.}\ (1, 1, 1) \to \hat{1}^{>} = R^{-1}$. In this way the nonsymmetric component of the genometric is "compensated" by the corresponding inverse nonsymmetric component of the unit, thus preserving the original geometric axioms.

The above occurrence confirms that the basic invariant for isotopic, genotopic and, as we shall see in the next section, hyperstructural methods is given by

$$\text{length} \times \text{unit} \equiv \text{isolength} \times \text{isounit} \equiv \text{genolength} \times \text{genounit} . \qquad (3.$$

Note that, as indicated in the preceding subsection, the formulation of genospaces with nonsymmetric metrics over a conventional field, would yield gross inconsistencies, such to void any possibility of consistent applications.

The axiomatic characterization of irreversibility via the genotopic methods now begins to emerge. In fact, conventional and isotopic spaces are structurally *reversible* because their metric is *symmetric,* thus having no established direction in time. On the contrary, genometric have a time direction by conception and realization precisely because of the loss of the symmetric character.

Lemma 3.2 [9]: *An axiomatization of irreversibility in interior gravitation is provided by inequivalent deformations of the modular action to the right from that to the left and of related metrics under a joint lifting of the unit per each action characterized by the inverses of said deformations.*

The last important genotopies which need to be indicated as basic methods are those of the differential and isodifferential calculus, resulting in the

with the seven-dimensional *total forward genounit*

$$\hat{1}^{>}_{tot} = \hat{1}_t^{>} \times \hat{1}^{>}_r \times \hat{1}^{>}_v = R_t^{-1} \times R^{-1} \times R^{-1} \tag{3.18}$$

where R_t a scalar and R is a real-valued nonsymmetric, nowhere degenerate 3×3 matrix and we have assumed for simplicity $\hat{1}^{>}_r \equiv \hat{1}^{>}_v$.

To characterize a systems one must: *first* select the forward genounit of time and of space; *second*, construct the total forward carrier space $\hat{S}^{>}$; and, *third,* formulate the theory with the forward genonewton equations.

It is easy to see that the forward genonewton equations are derivable from the first-order *forward genoaction*

$$\hat{A}^{>} = \int_{t_1}^{t_2} \hat{L}^{>}(\hat{t}^{>},\hat{r}^{>},\hat{v}^{>}) , \tag{3.19}$$

where the *forward genolagrangian* is a conventional form only properly written in forward genospaces, that is, with all products restricted to the selected ordering of the genotopy.

Variation of the genoaction then yields the following *genolagrange equations,* introduced here apparently for the first time,

$$\frac{\hat{d}^{>}}{\hat{d}^{>}} \frac{\hat{\partial}^{>}\hat{L}^{>}(\hat{t}^{>},\hat{r}^{>},\hat{v}^{>})}{\hat{\partial}\,\hat{v}^{>k}} - \frac{\hat{\partial}^{>}\hat{L}^{>}(\hat{t}^{>},\hat{r}^{>},\hat{v}^{>})}{\hat{\partial}^{>}\,\hat{r}^{k}_{.}} , \tag{3.20}$$

which must also be defined, for consistency, on the total forward genospace.

The genocanonical formulations are defined on the total genospace

$$\hat{S}^{>}(\hat{t}^{>},\hat{r}^{>},\hat{v}^{>}) = \hat{E}^{>}(\hat{t}^{>},\hat{R}_t^{>}) \times \hat{E}^{>}(\hat{r}^{>},\hat{\delta}^{>},\hat{R}^{>}) \times \hat{E}(\hat{p}^{>},\hat{\delta}^{>},\hat{R}^{>}) , \tag{3.21}$$

with the seven-dimensional *total forward genounit*

$$\hat{1}^{>}_{tot} = \hat{1}_t^{>} \times \hat{1}^{>}_r \times \hat{1}^{>}_p \equiv R_t^{-1} \times R^{-1} \times R \tag{3.22}$$

$$(\omega_{\mu\nu}) = \left(\frac{\partial^{>}\hat{R}^{>}_{\nu}}{\partial \hat{b}^{\mu}} - \frac{\partial^{>}R^{>}_{\mu}}{\partial \hat{b}^{\nu}} \right) = \begin{pmatrix} 0_{N\times N} & -I_{N\times N} \\ I_{N\times N} & 0_{N\times N} \end{pmatrix}, \tag{3.28a}$$

$$(\omega^{\alpha\beta}) = \left(\frac{\partial^{>}\hat{R}^{>}_{\nu}}{\partial \hat{b}^{\mu>}} - \frac{\partial^{>}R^{>}_{\mu}}{\partial \hat{b}^{\nu>}} \right)^{-1} = \begin{pmatrix} 0_{N\times N} & I_{N\times N} \\ -I_{N\times N} & 0_{N\times N} \end{pmatrix}, \tag{3.28b}$$

are the *conventional* covariant (symplectic) and contravariant (Lie) tensors, respectively, which hold in view of the properties originating from the genodifferential calculus

$$\partial^{>}\hat{R}^{>}_{\nu} / \partial \hat{b}^{\mu>} \equiv \partial R^{\circ}_{\nu} / \partial b^{\mu}, \quad R^{\circ} = (p, 0), b) (r, p). \tag{3.29}$$

The preservation of the conventional symplectic and Lie structures under genotopies has truly far reaching implications, on both grounds of pure and applied mathematics. In fact, it confirms the existence of a consistent step–by–step genotopic lifting of the entire body of isotopic methods, including the isosymplectic geometry, and the Lie–Santilli isotheory (that is, including universal isoassociative algebras, isoalgebras, isogroups, isosymmetries, isorepresentation theory, etc.) [9,10]. In turn, this occurrence ensures the axiomatic consistency of the isotopic theories, thus its availability for consistent applications.

It is evident that we cannot possibly study these properties in the necessary details here. We merely limit ourselves to indicate that the preservation of the symplectic structure under genotopies ensures the existence of a consistent, unique and unambiguous *genosymplectic quantization.*

The latter results can be illustrated via the *naive genoquantization*

$$\hat{A}^{>}(\hat{t}^{>} \hat{r}^{>} \quad \to \quad -i\hat{1}^{>}(\hat{t}^{>}, \hat{r}^{>}, \hat{\psi}^{>}, ...) L \hat{\psi}^{>}, \tag{3.30}$$

under which we have the *forward genoschrödinger equations* for the Hamiltonian and for the momentum, respectively,

$$i \frac{\partial^{>} \hat{\psi}^{>}(\hat{t}^{>}, \hat{r}^{>})}{\partial \hat{t}^{>}} = i\hat{1}_{t}^{>} \frac{\partial \hat{\psi}^{>}}{\partial t} = \hat{H}^{>} > \hat{\psi}^{>}(\hat{t}^{>}, \hat{r}^{>}) = \hat{H}^{>} \times R \times \hat{\psi}^{>} \tag{3.31a}$$

The emerging operator theory, called *genotopies of quantum mechanics,* and known as the *Lie-admissible branch of hadronic mechanics* [9,10], are characterized by the following basic mathematical structures:

1) The *forward genoassociative envelope* with operators $\hat{A}$, $\hat{B}$, ..., genounit $\hat{1}^{>} = R^{-1}$ and genoproduct $\hat{A}>\hat{B}$;

2) The *forward genofields* $\hat{C}^{>}(\hat{c}^{>},+,>)$ and $\hat{R}^{>}(\hat{n}^{>},+,>)$ of *forward genocomplex numbers* $\hat{c}^{>}$ and *forward genoreal numbers,* respectively;

3) The *forward genoeuclidean spaces* $\hat{E}^{>}(\hat{t}^{>},\hat{R}_t^{>})\times\hat{E}^{>}(\hat{r}^{>},\hat{\delta}^{>},\hat{R}^{>})\times\hat{E}^{>}(\hat{p}^{>},\hat{\delta}^{>},\hat{R}^{>})$ as the basic carrier space and the *genohilbert space* $\hat{\mathcal{H}}^{>}$ with *forward genostates* $\hat{\psi}^{>}(\hat{t}^{>},\hat{r}^{>})$, $\hat{\phi}^{>}(\hat{t}^{>},\hat{r}^{>})$, ... and *forward genoinner product*

$$\hat{\mathcal{H}}^{>}: \quad (\hat{\psi},\hat{\Phi})^{>} = R^{-1}\times\int d^3r\,\hat{\psi}^{\dagger}(r)\times R\times\phi(r) \in \hat{C}^{>}, \tag{3.35}$$

plus the genotopies of all remaining aspects of the isotopies, such as isoexpectation values, isoeigenvalues, etc., as well as with the backward version of all preceding formulations (see again volumes [9,10] for brevity).

The above genotopies of quantum mechanics preserve all the conventional axiomatic properties, including Hermiticity, thus observability, etc., to such an extend that the genooperator and conventional formulations coincide at the abstract level. This property evidently assures the axiomatic consistency of the theory (see [10] for details).

The fundamental difference in applications is that the Hamiltonian is always conserved for quantum mechanics,

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Santilli [6] therefore first modified Albert's definition of Lie-admissibility by adding the condition that U should contain a Lie algebra as a particular case, and then generalized the definition further to an algebra U such that U^{-} is Lie-isotopic (rather than Lie) and the algebra U admits a Lie-isotopic algebra as a particular case. The latter condition is today referred to as the *Lie-Santilli Lie-admissibility*. The abstract product ab in the realization $(a, b) = a<b - b>a = a\times R\times b - b\times S\times a$, where $a\times R$, $R\times b$, etc., are associative products, verify the conditions for Lie-Santilli admissibility. In fact, $ab = (a, b)$ is nonassociative, it admits as a particular case the Lie-Santilli isoalgebra for $R = S$, and the attached antisymmetric algebra is Lie-Santilli, rather than Lie, $[a, b]_U = (a, b) - (b, a) = a\times T\times b - b\times T\times a$, $T = R + S$.

and particular form for the energy

$$i\, dH_0 / dt = - i\, \gamma\, H_0, \tag{3.43}$$

thus providing the desired representation of dissipation.

We therefore have the following

Lemma 3.3: *An axiomatization of irreversibility via classical or quantum dynamical equations is provided by the genotopies of conventional equations.*

In fact, the dynamical equations of this section are *structurally irreversible,* that is, they are irreversible irrespective of the behaviour under time–reversal of the Hamiltonian as well as of the isounit. To be more specific in this point (which is rather fundamental for applications), the genotopic equations are irreversible even when the Hamiltonian and the genounit do not dependent explicitly on time, thus being time–reversal invariant.

We close this section by indicating the recent emergence of very serious problematic aspects of other formulations of open *nonconservative* systems, such as their representation via the addition of an "imaginary potential iV to the kinetic energy K in the Hamiltonian, H = K +iB, with dynamical equations[10]

$$i\, dQ / dt = Q \odot H = Q \times H - H^{\dagger} \times Q, \qquad H^{\dagger} = K - i\, V \neq H = K + iV, \tag{3.44}$$

[10] The reader should be warned that these equations have no connection with the reality to be represented. In fact, the dissipation is due to forces which are of contact *nonpotential* type and their representation via a potential of any nature can be easily proved to represent a totally different system. A definite illustration is given by the trajectory of a space–ship during re–entry in atmosphere which dissipates its kinetic energy. The representation of such a dissipation via an "imaginary potential" in the Hamiltonian can be easily proved to describe a trajectory which has no connection of any kind with the actual trajectory of the space–ship (see monographs [4,5] for detailed studies of nonpotential forces).

physical effects.

Even though novel in theoretical biology, the notion occurs in a number of instances in physics, when studied at sufficient depth. The first example is given by the familiar *Pauli exclusion principle* according to which two electrons in the same state can "exclude each other", i.e., can force each other to have different quantum numbers. After a state has been filled up, additional electrons are repelled.

It is well know in quantum mechanics that the above mutual exclusion positively cannot be represented with any known interactions carrying a potential (because that would dramatically alter the experimentally established spectral lines). But the Pauli exclusion principle is precisely a case of *isotopic correlation*, that is, correlation represented via the isounit, and essentially due to the mutual wave overlapping of the wavepackets of the electrons which, as worked out in the preceding subsection, carries no potential energy of any kind (see [10] for details).

Pauli exclusion principle therefore illustrates the existence of a *correlation without any potential energy*, in which there is no energy consideration of any kind. Yet the end results are specific and concrete (the exclusion of certain quantum number or the expulsion of additional electrons in a saturated state).

Another example in particle physics is the *Bose–Einstein correlation* which has been experimentally detected in the proton–antiproton annihilation at both high and low energies. It may be recommendable to briefly outline the latter correlation here because its reformulation in theoretical biology is straightforward.

In the Bose–Einstein correlation, the $p-\overline{p}$ first fuse together into a state called the *fireball*, which then decays rapidly into various particles whose end results are mesons (obeying the Bose–Einstein statistics) which, even though at large distances, are correlated (for a review of conventional studies see, e.g., [47] and for its isotopic study see [48]).

The reasons why generalized methods are expected to play a central role for the Bose–Einstein correlation are multifold. First, there is a rather general consensus in the scientific community that *local–differential theories cannot have correlation.* In fact, if the fireball is made up of a finite set of isolated points, the final mesons simply cannot be correlated. This rules out *ab initio* the use of quantum mechanics as *exactly* applicable in favor of hadronic mechanics which

$$\hat{C}_n = \overbrace{<1,a|<1,b|.....<n,a|<n,b|}\ (\hat{T}) \begin{pmatrix} |1,a> \\ |1,b> \\ ..... \\ |n,a> \\ |n,b> \end{pmatrix} =$$

$$= \sum_{k,i.j.} (<k,a|\hat{T}_{kk}|k,a> + <k,b|\hat{T}_{kk}|k,b> + <i,a|\hat{T}_{ij}|j,b>) \qquad (5.59)$$

and exhibits precisely the cross terms needed for a description of correlation from first axiomatic principles.

In current "semiphenomenological models", these cross terms are introduced via a number of artificial machinations. However, for scientific objectivity we must admit that these models are, strictly speaking, beyond the axiom of expectation value, thus confirming the inability of quantum mechanics to derive the event in an *exact* form from first principles.

A systematic study of the correlation via the use of *relativistic hadronic mechanics* on isominkowskian spaces [10] has been conducted in memoir [48] of which we can only outline here the main lines. First, one can see that the representation of the correlation requires a *nondiagonal isotopic element.* Various arguments derived from experimental data then yield the structure

$$\hat{T} = S \times \mathcal{F} \times \begin{pmatrix} K_{a1} & K_{a2}(1 - \exp \int d^4x\, \psi^\dagger_{b2}\, \psi_{a1}) \\ K_{b1}(1 - \exp \int d^4x\, \psi^\dagger_{a2}\, \psi_{b1}) & K_{b2} \end{pmatrix} \qquad (5.60)$$

where the K's are parameters, the ψ's are the wave functions, S is a *shape factor,* that is, a term representing the *extended, nonspherical and deformable shape of the fireball,*

$$S = \text{diag.}(S_{\mu\mu}) = \text{diag.}(b_1^2, b_2^2, b_3^2, b_4^2), \qquad (5.61).$$

with b_1^2, b_2^2, b_3^2, representing the semiaxes of the fireball and b_4^2 representing its density, and $\mathcal{F}$ is a nonlocal-integral terms representing nonpotential interactions,

1) Direct representation of the nonlocal origin of the correlation via the metric of the underlying geometry;

2) Direct representation of the fireball as an extended, nonspherical and deformable entity;

3) Prediction of the maximal and minimal limits of the two-point

EXPERIMENTAL VERIFICATION OF THE BOSE–EINSTEIN ISOCORRELATION FUNCTION

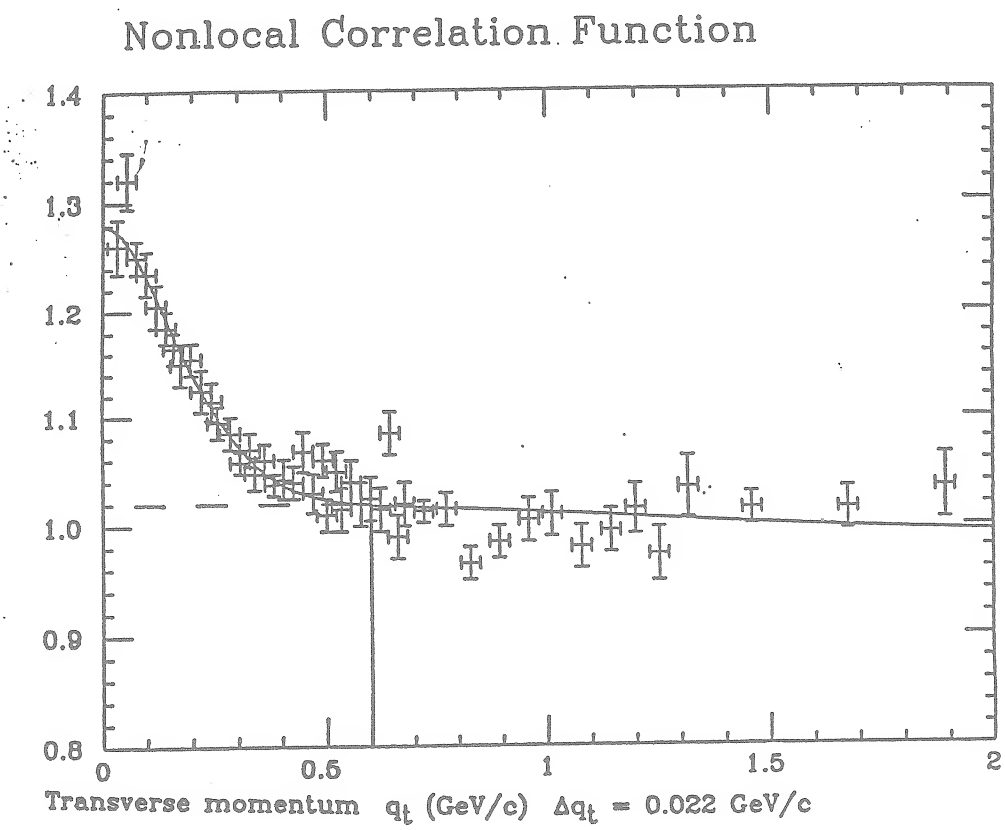

FIGURE 5.7. A reproduction of the main plot done by Mignani and Cardone [49] on Santilli's isocorrelation function [48] via the use of the experimental data from the UA1 experiments at CERN. In particular, the plot identified the numerical values of the isotopic element according to the expression for the b's (which have been

5.5. Additional miscellaneous applications.

We close this monograph with the mere *indication* of the *possibilities* of the isotopic, genotopic and hyperstructural methods in theoretical biology, with particular reference to some of the contributions in this volume [50].

The general rule is that the isotopies should be applicable only under sufficient evidence of nonlinear, nonlocal nonhamiltonian effects, correlation or other characteristics which are outside the representational capability of a Lagrangian or a Hamiltonian, e.g., the representation of shapes, deformations, etc.

Also as a general rule, the reader should not expect drastic changes of existing results, but only the identification of possible, generally small new contributions and related effects.

Lifting of Hehrenberg-Pavlov studies in [50]. Here the isotopies can reduce the inhibition of plasmid duplication to primitive nonlocal effects; the differential calculus used in the excellent macroscopic description of plasmids number control can be lifted into the *genodifferential calculus* (Sect. 3) which naturally provides an axiomatization of the irreversibility of the inhibition while incorporating the origin of the control itself in the genounit; finally the entire results can be re–interpreted via the isogeometries with suggestive novel notions.

Lifting of Tributsch-Pohlmann studies in [50]. The latter studies deal with classical synergetic mechanisms associated with the absorption and utilization of photon energy based on microscopic *dissipative* processes. As such, the studies are ideally set for their quantitative representation via the genoanalytic equations in their classical profile, as a collection of genooperator counterparts for their microscopic origin, yielding an intrinsic representation of the irreversibility of the process, as well as a structural representation of the photon absorption, all done via the generalized *unit* of the theory.

Lifting of Pohlmann-Tributsch studies in [50]. In this case the isotopies of

Lifting of Tributsch studies in [50]. The reader should be aware that the writing of this memoir was stimulated by a lucid talk on the need for the energy to have an "arrow of time" delivered by Prof. Tributsch at the *First International Workshop on New Frontiers in Theoretical Biology* held at the Castle Prince Pignatelli in August 1995, and by the following discussions. The entire genotopic formulations of this memoir have been written for the specific purpose of providing an axiomatic formulation of Prof. Tributsch view, that is, a formulation which preserve all axiomatic properties of classical and quantum mechanics despite the presence of an arrow of time, and it is realized, e.g., via the genohamilton equations (3.26) at the classical level, or the genoschrödinger equations (3.51) at the operator level.

Additional liftings can be identified by the interested reader. Further comments on isotopies are presented by Illert's contribution in [50].

A.2: Isopythagorean Theorem

Consider a conventional two–dimensional Euclidean space $E = E(r,\delta,R)$ with contravariant coordinates $r = \{r^k\} = \{x, y\}$ and metric $\delta = \text{diag.}\ (1, 1)$ over the field $R = R(n,+,\times)$ of real numbers n with conventional sum + and multiplication × and respective additive unit 0 and multiplicative unit 1. The fundamental notion of this space is the assumption of the basic unit $I = \text{diag.}\ (1, 1)$ which implies the assumption of the same basic (dimensionless) unit +1 for both x– and y–axes, resulting in the familiar *Euclidean distance* among two points $x, y \in E$

$$D = [(x_1 - x_2)(x_1 - x_2) + (y_1 - y_2)(y_1 - y_2)]^{1/2} \in R(n,+,\times). \qquad (A.1)$$

The quantity $D^2 = D \times D$, $\times \in R$, then represents the celebrated *Pythagorean theorem* expressing the hypothenuse D of a right triangle with sides A and B according to the familiar law $D^2 = A^2 + B^2$.

The flat geometry of the plane $E(r,\delta,R)$ permits the introduction of the trigonometric notion of "angle α" between two intersecting straight vectors, and of "cosinus of α" which, for the case when the vectors initiate at the origin $0 \in E$ and go to two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, is given by

$$\cos\alpha = \frac{x_1 x_2 + y_1 y_2}{(x_1 x_1 + y_1 y_2)^{1/2}(x_2 x_2 + y_2 y_2)^{1/2}} . \qquad (A.2)$$

From the above definition one can derive the entire conventional trigonometry. For instance, by assuming that the points are on a circle of unit radius, D = 1, for $P_1(x_1, y_1)$ and $P_2(1, 0)$ we have $\cos\alpha = x_1$, for $P_1(x_1, y_1)$ and $P_2(0, 1)$ we have $\sin\alpha = y_1$, with consequential familiar properties, such as $\sin^2\alpha + \cos^2\alpha = 1$, etc.

Consider now the *two–dimensional isoeuclidean space of Class I,* $\hat{E} = \hat{E}(\hat{r},\hat{\delta},\hat{R})$ (Sect. I.3.3) over the isofield $\hat{R} = \hat{R}(\hat{n},+,\hat{\times})$ of isoreal numbers $\hat{n} = n\hat{\times}\hat{1}$, where the isounit $\hat{1}$ is a positive–definite 2×2–matrix whose elements have a well behaved but otherwise arbitrary dependence on time t, the local coordinates r and

with projection in the conventional plane $E(r,\delta,R)$

$$\hat{D}^{\hat{2}} = [\, A\, b_1^{\,2}(t, r, \dot{r}, ...)\, A \;+\; B\, b_2^{\,2}(t, r, \dot{r}, ...)\, B\,] \times \hat{1}\,, \qquad (5.C.6)$$

that is, the isosquare of the isohypothenuse of an isoright isotriangle is the sum of the isosquare of the isosides.

To understand the geometric meaning of the above theorem, we recall that all isotopic notions have, in general, *three* different interpretations, the first in isospace $\hat{E}(\hat{r},\hat{\delta},\hat{R})$, the second via the projection in the original space $E(r,\delta,R)$, and the third in a *conventional* Euclidean space $E(\bar{r},\delta,R)$ over the conventional reals $R(n,+,\times)$ whose interval coincides with that in isospace. The latter condition is easily verified by the assumption

$$\bar{x} = \hat{x}\, b_1(t, x, y, \dot{x}, \dot{y}, ...)\,, \qquad \bar{y} = y\, b_2(t, x, y, \dot{x}, \dot{y}, ...)\,, \qquad (A.7)$$

under which

$$[\,(x_1 - x_2)\, b_1^{\,2}\,(x_1 - x_2) \;+\; (y_1 - y_2)\, b_2^{\,2}\,(y_1 - y_2)\,]^{1/2} \equiv$$

$$\equiv [\,(\bar{x}_1 - \bar{x}_2)(\bar{x}_1 - \bar{x}_2) \;+\; (\bar{y}_1 - \bar{y}_2)(\bar{y}_1 - \bar{y}_2)\,]^{1/2}. \qquad (A.8)$$

The properties in isospace follow the general rules of all isotopies, that is, the preservation of all original properties, including their numerical values. Thus, straight lines in conventional space are mapped into *isostraight isolines* in isospace, i.e., lines which coincide with their tangent when computed in isospace; perpendicular lines in conventional space are mapped into *isoperpendicular isolines* whose angle is indeed 90° when measured in isospace, that is, with respect to its own isounit (see below); etc.

In this sense, a right triangle in the conventional plane remains so in isoplane, and the conventional Pythagorean Theorem holds also in isospace.

To understand the remaining geometric meaning we also have to consider

lines which are *straight* in $\hat{E}(\hat{r},\delta,\hat{R})$ become *curved* in $E(\bar{x},\delta,R)$, according to the rule:

$$\hat{a} \hat{\times} \hat{x} + \hat{b} \hat{\times} \hat{y} + \hat{c} = 0 \rightarrow$$

$$\rightarrow a \bar{x} b_1^{-1}(t, x, y, ...) + b \bar{y} b_2^{-1}(t, x, y, ...) = 0, \quad \hat{a}, \hat{b}, \hat{c} \in \hat{R} . \qquad (A.9)$$

The projection of the Isopythagorean Theorem in a conventional plane then results in the map of a right triangle into a geometric figure in which the sides are curved, with one intersection per pair as in Figure A.1.

A.3: Isotrigonometric functions

Let us use again the convention according to which the symbols $\hat{a}$, $\hat{x}$, $\hat{y}$, etc., denote quantities computed in isospace $\hat{E}(\hat{r},\delta,\hat{R})$, the symbols $\bar{a}$, $\bar{x}$, $\bar{y}$, etc., denote the corresponding quantities when computed in the plane $E(\bar{r},\delta,R)$, and the symbols a, x, y, etc., denote the projection in the conventional space $E(r,\delta,R)$.

Suppose that the two points $\hat{P}_1(\hat{x}_1, \hat{y}_1)$ and $\hat{P}_2(\hat{x}_2, \hat{y}_2)$ represent isostraight *isovectors* initiating from the origin $\hat{0} \in \hat{E}(\hat{r},\delta,\hat{R})$. Let us denote with $\hat{\alpha}$ the *isoangle* between these two isovectors to be identified below. Consider their *identical* reformulation in the conventional space $E(\bar{x},\delta,R)$, in which case the angle $\hat{\alpha}$ persists. We can then introduce the conventional $\cos \hat{\alpha}$ in $\hat{E}(\hat{r},\delta,\hat{R})$

$$\cos \hat{\alpha} = \frac{\bar{x}_1 \bar{x}_2 + \bar{y}_1 \bar{y}_2}{(\bar{x}_1 \bar{x}_1 + \bar{y}_1 \bar{y}_2)^{1/2} (\bar{x}_2 \bar{x}_2 + \bar{y}_2 \bar{y}_2)^{1/2}} . \qquad (A.10)$$

with projection in $E(r,\delta,R)$

$$\cos \hat{\alpha} = \frac{x_1 b_1^2 x_2 + y_1 b_2^2 y_2}{(x_1 b_1^2 x_1 + y_1 b_2^2 y_1)^{1/2} (x_2 b_1^2 x_2 + y_2 b_2^2 y_2)^{1/2}} \qquad (A.11)$$

We now assume that the two points $\hat{P}_1(\hat{x}_1, \hat{y}_1)$ and $\hat{P}_2(\hat{x}_2, \hat{y}_2)$ are on the *unit isocircle*

$$= \cos^2 \hat{\alpha} + \sin^2 \hat{\alpha} = 1, \tag{A.16}$$

and general rules for an isosquare isotriangle with isosides $\hat{A}$ *and* $\hat{B}$ *and isohypothenuse* $\hat{D}$ *as in Diag. (a) of Fig. A.1*

$$\hat{A} = \hat{D}\,\text{isocos}\,\hat{\gamma}\,, \quad \hat{B} = \hat{D}\,\text{isosin}\,\hat{\gamma}\,, \quad \hat{A}/\hat{B} = \text{isotan}\,\hat{\gamma}, \text{ etc.} \tag{A.17}$$

The isoangles have been identified from the representation theory of isorotations in a plane (see Vol. II, Ch. 6, ref. [10]), and results to be given by

$$b_1\, b_2\, \alpha = \hat{\alpha}\,. \tag{A.18}$$

where the factor $b_1 b_2$ *is fixed for all possible isoangles of a given isoeuclidean space.* This means that the *isotopy of the trigonometric angles* is given by

$$\alpha \quad \rightarrow \quad b_1\, b_2\, \alpha = \hat{\alpha}\,, \tag{A.19}$$

with consequential *angular isotopic element*

$$\hat{T}_{\hat{\alpha}} = b_1\, b_2 = (\text{Det}\,\hat{T})^{1/2} \tag{A.20}$$

and *angular isounit*

$$\hat{1}_{\hat{\alpha}} = b_1^{-1}\, b_2^{-2} = (\text{Det.}\,\hat{1})^{1/2} \tag{A.21}$$

where $\hat{T}$ and $\hat{1}$ are the isotopic element and isounit, respectively, of the isoeuclidean plane, Eq.s (3).

Isoangles $\hat{\alpha}$ have a nonlinear and integro-differential dependence on the local coordinates and their derivatives when projected in the original Euclidean plane with expression

$$\text{isosin}\,\hat{\alpha} + \text{isosin}\,\hat{\beta} = 2\,b_1^{-1}\,\text{isosin}\,\tfrac{1}{2}(\hat{\alpha} + \hat{\beta})\,\text{isocos}\,\tfrac{1}{2}(\hat{\alpha} - \hat{\beta})\,. \qquad \text{(A.25d)}$$

The interested reader can then work out the isotopies of other trigonometric properties.

We are now equipped to introduce the following

Definition A.2 [44]: *The "isopolar coordinates" are the polar coordinates of the unit isocircle in the isoeuclidean plane* $\hat{E}(\hat{r},\hat{\delta},\hat{R})$, *and can be written*

$$\hat{x} = \text{isocos}\,\hat{\alpha}\,, \qquad \hat{y} = \text{isosin}\,\hat{\alpha}\,, \qquad \text{(A.26)}$$

with projection in the conventional Euclidean plane $E(r,\delta,R)$

$$x = b_1^{-1}\cos(b_1 b_2 \alpha)\,, \qquad \hat{y} = b_2^{-1}\,\text{isosin}(b_1 b_2 \alpha)\,. \qquad \text{(A.27)}$$

and property

$$\hat{x}^{\hat{2}} + \hat{y}^{\hat{2}} = x\,b_1^2\,x + y\,b_2^2\,y =$$

$$= b_1^2\,\text{isocos}^2\,\hat{\alpha} + b_2^2\,\text{isosin}^2\,\hat{\alpha} = \cos^2\hat{\alpha} + \sin^2\hat{\alpha} = 1\,. \qquad \text{(A.28)}$$

The exponential formulation of trigonometric functions also admits a simple, yet unique and effective isotopic image. It requires the lifting of the conventional enveloping associative algebras ξ and their infinite-dimensional basis with conventional unit I and product × (the Poincaré-Birkhoff-Witt Theorem) into the *enveloping isoassociative algebras* $\hat{\xi}$ of Sect. I.4.3 with isotopic image of the original infinite basis characterized by the isounit $\hat{1}$ and the isotopic product $\hat{\times} = \times\hat{T}\times$ (the *isotopic Poincaré-Birkhoff-Witt Theorem*).

The isotrigonometric functions can then be expressed in term of the *isoexponentiation* according to the rule

However, the projection of the above structure back to the conventional plane implies the deformation of the circle into the ellipse (Diag. (b)), with deformation of the polar coordinates

$$x = \cos \alpha \quad \rightarrow \quad x = b_1^{-1} \cos (b_1 b_2 \alpha),$$

$$y = \sin \hat{\alpha} \quad \rightarrow \quad \hat{y} = b_2^{-1} \sin (b_1 b_2 \alpha).$$

The reader is warned *not* to attempt the computation of *isotrigonometric* properties in the *conventional* Euclidean plane. This is due to the fact that the $\hat{x}$ and $\hat{y}$ isostraight axes in $\hat{E}$ are mapped into *curves* in E, as depicted in Diag. (b). Mathematical consistency of the isotrigonometry is then achieved only in isospace.

The interested reader can then work out additional properties of the isotrigonometric functions.

A.4: Isohyperbolic functions

The application of the preceding method to the lifting of the hyperbolic functions is straighforward, leading to the following:

Definition A.3 [44,9]: *The "isohyperbolic functions" on isoeuclidean space* $\hat{E}(\hat{r},\hat{\delta},\hat{R})$ *of Class I are given by*

$$\text{isocosh } \hat{\alpha} = b_1^{-1} \cosh (b_1 b_2 \alpha), \tag{A.30a}$$

$$\text{isosinh } \hat{\alpha} = b_2^{-1} \sinh (b_1 b_2 \alpha), \tag{A.30b}$$

with basic property

$$b_1^2 \text{ isocosh}^2 \hat{\alpha} - b_2^2 \text{ isosinh}^2 \hat{\alpha} = 1, \tag{A.31}$$

and derivation via the isoexponentiation

A.5: Isoexponential, isologarithm and other elementary isofunctions

We close this appendix with the identification of a few additional special isofunctions.

Definition A.4 [9]: *Let* f(x) *be an ordinary function verifying the needed regularity and continuity conditions on a given closed interval of the real variable* $x \in R(n,+,\times)$. *Then the "isotopic image"* $\hat{f}(\hat{x})$ *of* f(x), *is a function of the corresponding closed isointerval of the isoreal number* $\hat{x} = x \times \hat{1} \in \hat{R}(\hat{n},+,*)$ *generally given by the rule*

$$\hat{f}(\hat{x}) = \hat{1} \times f(x) . \tag{A.35}$$

An example of *elementary isofunctions* is the *isopower* on the isofield $\hat{R}(\hat{n},+,\hat{\times})$

$$\hat{f}(\hat{x}) = \hat{x}^{\hat{n}} = \hat{x} \,\hat{\times}\, \hat{x} * ..... \,\hat{\times}\, \hat{x} \text{ (n-times)} = \hat{1} \times x^{n} . \tag{A.36}$$

A lesser trivial and most fundamental isofunction is the *isoexponentiation* which is the exponentiation in the isoenvelope $\hat{\xi}$ with isounit $\hat{1} = \hat{T}^{-1}$ and isoproduct $A \hat{\times} B = A \times \hat{T} \times B$. which is given by the infinite isoseries

$$\hat{e}^{A} = e_{\hat{\xi}}^{A} = \hat{1} + A / 1! + A^{\hat{\times}} / 1! + ... =$$

$$= \hat{1} \times \{ e^{\hat{T} \times A} \} = \{ e^{A \times \hat{T}} \} \times \hat{1} , \tag{A.37}$$

where e is the ordinary exponentiation.

It is evident that the isounitary transforms in isohilbert space are expressible interns of the above isoexponentiation, resulting in the time evolution law for operator isotopic methods (Sect. II)

diagonal isounits with imaginary terms). The hyperfunctions are completely unexplored at this writing.

The author hopes to have illustrated in this appendix once more that the removal of the current restriction of our entire mathematical knowledge to the trivial unit identified since biblical times, and the use of structurally more general units, implies a rather vast broadening of all of mathematics, beginning with the most elementary ones such as angles, and then following with all remaining structures, permitting basically novel applications in a variety of fields [10].

dimensional isoeuclidean space $\hat{E}(r,\hat{\delta},\hat{R})$ with isometric

$$\hat{\delta} = T\,\delta, \quad \delta = \text{diag.}\,(1,\,1,\,1), \quad T = \text{diag.}\,(T_x,\,T_y,\,T_z)\,. \tag{B.3}$$

Introduce the redefinitions of the Cartesian coordinates

$$\bar{x}_1 = x\,T_x^{\frac{1}{2}}, \qquad \bar{y} = y\,T_y^{\frac{1}{2}}, \qquad \bar{z} = z\,T_z^{\frac{1}{2}}, \tag{B.4}$$

which are such to reduce the isoinvariant in $\hat{E}(r,\hat{\delta},\hat{R})$ into an identical form in a conventional Euclidean space $E(\bar{r},\delta,R) \neq E(r,\delta,R)$,

$$\hat{r^2} = x\,T_x\,x + y\,T_y\,y + z\,T_z\,z \equiv \bar{x}\,\bar{x} + \bar{y}\,\bar{y} + \bar{z}\,\bar{z} = \bar{r}^2\,. \tag{B.5}$$

Next, we introduce the *isospherical angles*

$$\hat{\theta} = T_z^{\frac{1}{2}}\,\theta, \qquad \hat{\phi} = T_x^{\frac{1}{2}}\,T_y^{\frac{1}{2}}\,\phi\,, \tag{B.6}$$

defined to coincide with the original angles prior to the deformation. They are derivable from the representation theory of the isorotational group $\hat{O}(3)$ which is here omitted for brevity [10].

Under these assumptions, the *isospherical coordinates* can be first written in the form [10] (see Fig. B.1)

$$x = r\,T_x^{-\frac{1}{2}} \sin\left(T_z^{\frac{1}{2}}\,\theta\right) \cos\left(T_x^{\frac{1}{2}}\,T_y^{\frac{1}{2}}\,\phi\right) \tag{B.7a}$$

$$y = r\,T_y^{-\frac{1}{2}1} \sin\left(T_z^{\frac{1}{2}}\,\theta\right) \sin\left(T_x\,T_y\,\phi\right), \tag{B.7b}$$

$$z = r\,T_z^{-\frac{1}{2}} \cos\left(T_z^{\frac{1}{2}}\,\theta\right), \tag{B.7c}$$

We can then introduce the simplest possible form of the *isomeasure* on $\hat{E}(r,\hat{\delta},\hat{R})$, that in terms of conventional differentials with the isotopic element independent from the local variables

$$= T_y^{-\frac{1}{2}} \text{isocos}\,\hat{\phi} + i\, T_x^{-\frac{1}{2}} \text{isosin}\,\hat{\phi}\,, \tag{B.11a}$$

$$\hat{t} = T_x^{\frac{1}{2}} T_y^{\frac{1}{2}}\,, \qquad \hat{1}_\phi = \hat{t}^{-1}\,. \tag{B.11b}$$

The next issue is the appropriate isotrigonometric formulation of the remaining terms in $\hat{\theta}$. At this point there is the emergence of a further degree of freedom which is "hidden" in the *isotopic* theory itself and completely absent in quantum mechanics.

By inspecting structure (B.7) one could conclude that the isogauss plane for the polar angle has the isotopic element $T = \text{diag.}\,(T_z,\, 1)$. However, one can also introduce the following redefinition of the isotopic element in three-dimensional space

$$T_x^{\frac{1}{2}} = B_{22}\, B_{11}\,, \quad T_y^{\frac{1}{2}} = B_{22}\, B_{12}\,, \quad T_z^{\frac{1}{2}} = B_{21}\,, \tag{B.12a}$$

$$B_{21}\, B_{22} = B_{11}\, B_{12}\,. \tag{B.12b}$$

with solution

$$B_{11}{}^3 = T_x\, T_z^{\frac{1}{2}} / T_y^{\frac{1}{2}}\,, \tag{B.13a}$$

$$B_{12}{}^3 = T_y\, T_z^{\frac{1}{2}} / T_x^{\frac{1}{2}}\,, \tag{B.13b}$$

$$B_{22}{}^3 = b_1\, b_2 / b_3\,, \quad B_{21} = b_3, \tag{B.13c}$$

$$B_{11}{}^3\, B_{12}{}^3 = B_{22}{}^3\, B_{21}{}^3 = T_x^{\frac{1}{2}}\, T_y^{\frac{1}{2}}\, T_z\,, \tag{B.13d}$$

under which we can introduce the *general isospherical coordinates*

$$x = r\, \text{isosin}\,\hat{\theta}\ \text{isocos}\,\hat{\phi} =$$

$$= r\,[\, B_{22}{}^{-1} \sin(\, B_{21}\, B_{22}\, \theta\,)\,]\,[\, B_{11}{}^{-1} \cos(\, B_{11}\, B_{12}\, \phi)\,]\,, \tag{B.14a}$$

$$y = r\, \text{isosin}\,\hat{\theta}\ \text{isosin}\,\hat{\phi} =$$

$$T_\theta = B_{21} B_{22} \equiv T_\phi = B_{11} B_{12}, \tag{B.16a}$$

$$\hat{1}_\theta = B_{21}^{-1} B_{22}^{-1} \equiv T_\phi = B_{11}^{-1} B_{12}^{-1}, \tag{B.16b}$$

with evident computational advances.

The "hidden" isotopic degree of freedom in the transition from the decomposition

$$T = \text{diag}\,(T_x, T_y, T_z) = \text{diag}\,(T_x, T_y) \times \text{diag}\,(T_z, 1), \tag{B.17}$$

to the more general form underlying structure (5.5.14)

$$T = \text{diag}\,(T_x, T_y, T_z) = \text{diag.}\,(B_{11}^2, B_{12}^2) \times \text{diag}\,(B_{21}^2, B_{22}^2), \tag{B.18a}$$

$$T_x = B_{22}^2 B_{11}^2, \qquad T_y = B_{22}^2 B_{12}^2, \qquad T_z = B_{21}^2, \tag{B.18b}$$

is also important for numerous applications of the isotopies.

into all infinitely possible three-dimensional surfaces

$$\hat{r^2} = r^k \, T \, r^k = + r^1 \, T_x \, r^1 + r^2 \, T_y \, r^2 + r^3 \, T_z \, r^3 = \text{inv.} \qquad (C.3)$$

Some of the main properties of isorotations can then be expressed as follows:

Theorem C.1 [27,10]: *The isosymmetries* $\hat{O}(3)$ *of all infinitely possible deformations of the sphere on the isoeuclidean spaces* $\hat{E}(r,\hat{\delta},\hat{R})$, *verify the following properties:*

1) The groups $\hat{O}(3)$ *consist of infinitely many different simple groups corresponding to the infinitely many possible deformations of the sphere (explicit forms of the isometric);*

2) All isosymmetries $\hat{O}(3)$ *are locally isomorphic to* $O(3)$ *for positive-definite isounits or are isomorphic to the isodual* $O^d(3)$ *for negative-definite isounits; and*

3) The groups $\hat{O}(3)$ *constitute "isotopic coverings" of the conventional group* $O(3)$ *in the sense that:*

3.a) The groups $\hat{O}(3)$ *are constructed via methods (the Lie-Santilli theory* [6]*) structurally more general than those of* $O(3)$ *(the conventional Lie's theory);*

3.b) The groups $\hat{O}(3)$ *represent physical conditions (deformations of the sphere; inhomogeneous and anisotropic interior physical media; etc.) which are broader than those of the conventional symmetry (perfectly rigid sphere; homogeneous and isotropic space; etc.); and*

3.c) All groups $\hat{O}(3)$ *recover* $O(3)$ *identically whenever* $\hat{1} = I$ *and they can approximate the latter as close as desired for* $\hat{1} \approx I$.

It is generally believed in both the mathematical and physical literature that the rotational symmetry is broken by ellipsoidical deformations of the sphere. This belief is disproved by the Lie-Santilli theory because of the following:

Corollary C.1.A [loc. cit.]: *The rotational symmetry is not broken by ellipsoidical deformations of the sphere, but it is instead exact because of*

let us consider the three-dimensional isoeuclidean space $\hat{E}(r,\hat{\delta},\hat{R})$. Its local coordinates are usually assumed to be *contravariant* and we shall write $r = \{r^k\}$, $k = 1, 2, 3$. Assume that the isometric in its natural form has *covariant* indices

$$\hat{\delta} = (\hat{\delta}_{ij}\hat{}) = \text{diag.}\,(T_x, T_y, T_z)\,. \tag{C.5}$$

Its *contravariant* form is then given by

$$(\hat{\delta}^{ij}) = (\hat{\delta}_{ij})^{-1} = \text{diag.}\,(T_x^{-1}, T_y^{-1}, T_{zs}^{-1})\,,\ \hat{\delta}_{ik}\,\hat{\delta}^{kj} = \delta^i_j\,. \tag{C.6}$$

We consider now the *isophase space* $T^*\hat{E}(r,\hat{\delta},\hat{R})$ with local coordinates $a = \{a^\mu\} = \{r, p\} = \{k, p_k\}$, $\mu = 1, 2, ..., 6$, where the linear momentum p_k is *contravariant*, as usual. The raising and lowering of the indices therefore follows the rules

$$r_k = \hat{\delta}_{ki}\,r^i = T_k\,r^k\,, \qquad p^k = \hat{\delta}^{ki}\,p_i = T_k^{-1}\,p_k\ (\text{no sums})\,. \tag{C.7}$$

The classical Lie–Santilli brackets then assume the form[19]

$$[A\,\hat{,}\,B] = \frac{\partial A}{\partial r^p}\,\hat{1}^p_s\,\frac{\partial B}{\partial p_q} - \frac{\partial B}{\partial r^p}\hat{1}^p_q\,\frac{\partial A}{\partial p_q} =$$

$$= \frac{\partial A}{\partial r^k}\,T_k^{-1}\,\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial r^k}\,T_k^{-1}\,\frac{\partial A}{\partial p_k}\,, \tag{C.8}$$

To identify the Lie–Santilli algebra $\hat{so}(3)$, let us compute first the *classical fundamental isocommutation rules* which are readily given by[20]

$$([a^\mu\,\hat{,}\,a^\nu]) = \begin{pmatrix} [r^i\,\hat{,}\,r^j] & [r^i\,\hat{,}\,p_j] \\ [p_i\,\hat{,}\,r^j] & [p_i\,\hat{,}\,p_j] \end{pmatrix} = (\hat{\omega}^{\mu\nu}) = \begin{pmatrix} 0 & \hat{1} \\ -\hat{1} & 0 \end{pmatrix}, \tag{C.9}$$

---

[19] The proof that the above brackets do indeed verify the Lie axioms although in a generalized way is based on the isotopies of the symplectic geometry and, as such,. it cannot be reviewed here for brevity (see ref.s [[5] for brevity).

[20] These rules require the knowledge of the isoderivatives for which $\hat{\partial} r_i/\hat{\partial} r^j = \delta_{ij}$ [5].

$$[\, J^k \hat{,}\ p_i \,] = \epsilon^{kij}\, p_j, \tag{C.13b}$$

and evidently coincide with the conventional ones.

The desired *classical isorotational algebras* **sô(3)** are then given by [27][23]

$$\mathbf{s\hat{o}(3)}: \qquad [\, J^i \hat{,}\ J^j \,] = \epsilon^{ijk}\, J^k, \tag{C.14}$$

namely, the isocommutation rules of **sô(3)** have the same structure constants as those for the conventional **so(3).** This establishes the local isomorphism **sô(3) ~ so(3)** *ab initio.* For different classical realizations one may consult ref.s [9,10].

The *isocenter* of the enveloping algebra $\hat{\xi}$ is given by the isomagnitude of the isotopic angular momentum, $C^{(2)} = J^{\hat{2}}$, as expected. In fact,

$$[\, J^{\hat{2}} \hat{,}\ J^i \,] = 2\, \epsilon^{kij}\, J^k\, J^j \equiv 0. \tag{C.15}$$

The desired *classical isorotational group* SÔ(3) can then be expressed via the isoexponentiations in terms of the conventional generators and parameters (the Euler's angles)

$$\hat{\mathcal{R}}(\theta) = [\, \hat{\prod}_{k=1,2,3} e^{\theta_k \omega^{\mu\alpha} 1_{2\alpha}{}^{\nu} (\partial J_k / \partial a^\nu)(\partial / a^\mu)} \,] \hat{1} == \hat{S}(\theta)\, \hat{1}, \tag{C.16}$$

where the exponentials are expanded in the conventional associative envelope ξ for simplicity.

Note the true realization of the notion of isotopic lifting of a Lie symmetry, consisting of the preservation of the original generators and parameters of the symmetry, and the isotopic generalization of the structure of the Lie group itself.

The computation of examples is straightforward. For instance, a (classical)

[23] Isocommutation rules (6.1.14) disprove another popular belief in Lie's theory, that the compactness or noncompactness of an algebra can be ascertained from the structure constants. In fact, the structure constants ϵ^{ijk} are those of the *compact* **SO(3)** algebra, yet isoalgebra (6.1.14) can represent the *noncompact* **so(2.1)** algebra for T = diag. (1, 1, −1). The latter possibility has been *excluded* from the physical studies of this and of the following sections by restricted the isotopic element to be of Class I.

We therefore have the *general isorotation* on the isosphere

$$r' = \hat{\mathcal{R}}(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3) \hat{\times} r = \hat{\mathcal{R}}(\hat{\theta}_1) \hat{\times} \hat{\mathcal{R}}(\hat{\theta}_2) * \hat{\mathcal{R}}(\hat{\theta}_3) \hat{\times} r = \hat{S}(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)\, r\,,$$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} [\cos\hat{\theta}_1 \cos\hat{\theta}_3 - \sin\hat{\theta}_1 \cos\hat{\theta}_2 \sin\hat{\theta}_3] \\ [T_x^{\frac{1}{2}} T_y^{-\frac{1}{2}} (\sin\hat{\theta}_1 \cos\hat{\theta}_3 + \cos\hat{\theta}_1 \cos\hat{\theta}_2 \sin\hat{\theta}_3)] \\ [\sin\hat{\theta}_2 \sin\hat{\theta}_3] \end{pmatrix}$$

$$\begin{matrix} [-T_x^{-\frac{1}{2}} T_y^{\frac{1}{2}} (\cos\hat{\theta}_1 \sin\hat{\theta}_3 - \sin\hat{\theta}_1 \cos\hat{\theta}_2 \cos\hat{\theta}_3)] & [T_x^{-\frac{1}{2}} T_y^{\frac{1}{2}} \sin\hat{\theta}_1 \sin\hat{\theta}_2] \\ [-T_x^{-1} T_z^{\frac{1}{2}} (\sin\hat{\theta}_1 \sin\hat{\theta}_3 + \cos\hat{\theta}_1 \cos\hat{\theta}_2 \cos\hat{\theta}_3)] & [-T_y^{-\frac{1}{2}} T_z^{\frac{1}{2}} \cos\hat{\theta}_1 \sin\hat{\theta}_2] \\ [T_x^{\frac{1}{2}} T_y^{-\frac{1}{2}} \sin\hat{\theta}_2 \cos\hat{\theta}_3] & [\cos\hat{\theta}_2] \end{matrix} \Bigg) \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\hat{\theta}_1 = T_x^{\frac{1}{2}} T_y^{\frac{1}{2}} \theta_1\,, \quad \hat{\theta}_2 = T_z^{\frac{1}{2}} \theta_2\,, \quad \hat{\theta}_3 = T_x^{\frac{1}{2}} T_y^{\frac{1}{2}} \theta_3\,. \tag{C.22}$$

The *inverse general isorotation* is then given by

$$r = \hat{\mathcal{R}}(\pi - \hat{\theta}_3, \hat{\theta}_2, \pi - \hat{\theta}_1) * r'\,. \tag{C.23}$$

The representation of a decaying angular momentum is notoriously not possible with conventional rotations, but it is readily achieved by the isorotations with a functional dependence of the type $T_k^{\frac{1}{2}} = \exp\{-\gamma t\}$, under which we have

$$J^k = e^{-\gamma t} \epsilon^{kij} r^i p_j\,. \tag{C.24}$$

This illustrates the applicability of isorotations, not only for the invariance of the shape of the sea shells, but also their growth in time.

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4: ELEMENTS OF HYPERSTRUCTURAL METHODS

4.1: Introduction

Hyperstructures (see, e.g., [18–19]) are some of the most complex mathematical structures conceived by mathematicians until now. Part of their complexity is due to their virtually endless variety of formulations and realizations which evidently multiply the difficulties for their selection and realization into a form suitable for applications.

Yet, after due study of the problem, this author believes that the need for the hyperstructures in theoretical biology is simply unavoidable because the isotopic and genotopic methods are effective up to a certain complexity of the systems considered. As it is the fate of all quantitative sciences, theoretical biology will never admit a "final theory". The hyperstructures then emerge as the most natural avenue for broadening the relatively "simpler" isotopies and genotopies.

Moreover, this author believes that specific practical problems in theoretical biology will be invaluable in selecting the most effective hyperstructure among the large variety of possible ones. It is instructive here to recall that, rather than originating in mathematics and then propagating in the physics, the isotopic and genotopic methods reviewed in the preceding sections originated from specific physical problems and they have only marginally propagated into the mathematical literature until now. It is easy to predict a similar occurrence for the hyperstructures.

Along the latter lines, a fundamental condition has already been introduced

vastly unexplored, the expectation of a maturity similar to the preceding isotopies and genotopies not being realistic at this moment. But that's precisely the most interesting aspect for researchers with young minds of all ages.

4.2: Basic mathematical notions

A central mathematical property of the hyperstructures is that of being *multivalued,* that is, products which traditionally have only one value, may assume several different values.

As a simple illustration, all products among numbers considered until now, whether conventional $\times$, isotopic $\hat{\times}$ or genotopic $>$, $<$ admit one single value, e.g. $n \times m$ yields one single element of $R(n,+,\times)$. On the contrary, the hyperproduct of two numbers, here denoted $\odot$, implies multivaluedness, that is, $n \odot m$ yields a set of elements of $R(n,+,\times)$.

This central features of hyperstructure has a clear potential for new frontiers in theoretical biology, because it is particularly suited to represent, say, the birth of a new cell in which the original number of entities was *one* and the final number of entities is *two,* which is precisely a realization of the notion of multivaluedness.

The above multivaluedness has been reached in the contemporary literature in hyperstructures [17–19], including paper [16], via *hypermultiplications* while the unit remains scalar, that is, single–valued, when it exists.

However, all structural properties studied in this memoir are reducible to a primitive notion of generalized unit. In order to have a continuity of thought, in this memoir we introduce instead the *forward hyperunit* defined as a *finite and ordered set*

$$\{\hat{1}^{>}\} = \{\hat{1}_1^{>}(t, r, \dot{r}, \ddot{r}, ...), \hat{1}_2^{>}(t, r, \dot{r}, \ddot{r}, ...), ...\} \tag{4.1}$$

where the symbol {...} is that at times used in physics to denote a set, in which each element is a well behaved nowhere null structure such as a forward genounit (Sect. 3). The finite character of the set as well as the ordering of its

form. In fact, the hyperset is closed under sum and forward hyperproduct, and verifies the *conventional* associative and distributive laws. In short, under the above assumptions, all properties of fields, isofields and genofields are extendable to a multivalued structure *without* generalizing the original axioms, exactly as desired.

In summary, the essential character of the hyperstructures, that of expressing multivaluedness, can indeed be expressed in terms of the hyperunit, which is then reflected in the hypernumbers themselves. The multivaluedness of the operations is a mere consequence.

Note the insistence in preserving the conventional sum unchanged because its possible generalizations would imply divergences in exponentiation and other operations, thus prohibiting practical applications [15].

Once the fundamental structures of the unit, number and fields are identified, all remaining aspects are merely consequential and can be constructed essentially on ground of mutual compatibility.

We defined a *forward hypereuclidean space* the structure over the forward hyperfield of real hypernumbers, apparently introduced here for the first time,

$$\{\hat{E}^{>}\}\,(\{\hat{r}^{>}\},\{\hat{\delta}^{>}\},\{\hat{R}^{>}\})\,;\quad \{\hat{r}^{>}\} = \hat{r}^{>}\times\{\hat{1}^{>}\}\,,\quad \{\hat{\delta}^{>}\} = \{S\}\times\delta\,, \tag{4.8a}$$

$$\{\hat{r}^{\hat{2}>}\} = [\{\hat{r}^{>\,i}\}\{\hat{\delta}^{>}_{ij}\}\times\{\hat{r}^{>\,j}\}\times\{\hat{1}^{>}\}\in\{\hat{R}^{>}\} \tag{4.8b}$$

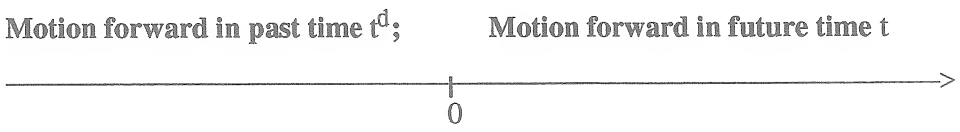
with a corresponding definition for *backward hypereuclidean space*. The *forward and backward hypereuclidean geometries* are the geometries of the above hyperspaces and can be constructed via a step-by-step hyperlifting of the isogeometries of Sect. 2 which is here left to the interested reader for brevity (but implied in the applications).

The property of Euclidean spaces which is preserved under hyperlifting is the quantity:

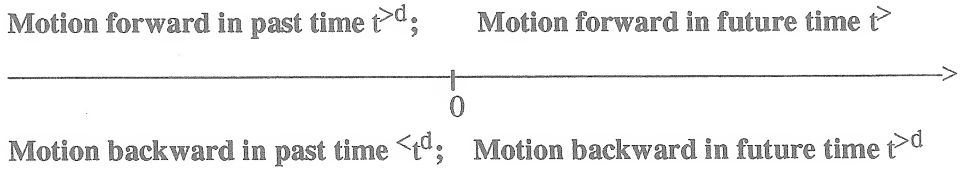
$$\text{Hyperlength}\times\text{Hyperunit} = \text{invariant}\,. \tag{4.9}$$

THE NEW NOTIONS OF TIME

ISOOTIME



GENOTIME



GENOTIME

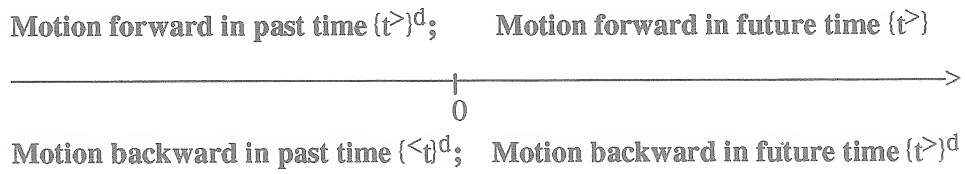


FIGURE 4.1. A summary view of the new notions of time suggested by the isotopic, genotopic and hyperstructural methods for biological structures. Eddington's "time arrows" are *four* and not two. In fact, we have *motion forward and backward in future and past times*. Their description therefore requires two different conjugations, one for the map of the future into the past and the other for the map of motion forward into that backward, each conjugation being bi–injective, that is, such that, when applied twice reproduces the original time. The map of the future into the past (or viceversa) is assumed in this monograph to be given by the

$$\textit{time isoduality} \quad t \quad \rightarrow \quad t^d = -t \,. \qquad (a)$$

with an arbitrary positive unit is equally admissible, provided that it is realized via the isotopic methods, i.e., via isofields, isospaces, etc. The isodual isotime is given by $\hat{t}^d = -\hat{t}$ and, as such, it is well defined. In particular, it also satisfies the geometric axiom (c) of *conventional* time

$$(\text{interval of time})^2 \times (\text{unit}) = [\,(t_2 - t_1) \times \hat{T}^d \times (t_2 - t_1)\,] \times \hat{1}^d = (t_2 - t_1)^2 \times 1.$$

We learn in this way that motion in past time is fully causal and compatible with our senses, provided that it is realized with our isodual isotopic methods, i.e., with the isodual isofields, isodual isospaces, etc. the *isoinverse isotime* generally coincides with the isotime, which renders the isotopic methods insufficient for sufficiently deep representation of irreversibility in biology.

The first axiomatic characterization of irreversibility is permitted by the *forward and backward genotimes* $\hat{t}^{>} = t \times \hat{1}^{>}(t, ...)$, $^{<}\hat{t} = t \times {}^{<}\hat{1}(t, ...)$, which characterize irreversible processes even when the genounits are not explicitly dependent on time, $\hat{1}^{>} \neq {}^{<}\hat{1}$. In this case both isoduality and isoinversions are well defined, with $\hat{1}^{>d} = -\hat{1}^{>}$, and $(\hat{1}^{>})^i = {}^{<}\hat{1}$ and the basic geometric property (c) is verified in each of the four arrows

$$[\,(t_2 - t_1) \times \hat{T}^{>} \times (t_2 - t_1)\,] \times \hat{1}^{>} = [\,(t_2 - t_1) \times {}^{<}\hat{T} \times (t_2 - t_1)\,] \times {}^{<}\hat{1} =$$

$$[\,(t_2 - t_1) \times \hat{T}^{>d} \times (t_2 - t_1)\,] \times \hat{1}^{>d} = [\,(t_2 - t_1) \times {}^{<}\hat{T}^d \times (t_2 - t_1)\,] \times {}^{<}\hat{1}^d =$$

$$= (t_2 - t_1)^2 \times 1,$$

This implies that all four arrows of time are possible in biological structures because compatible with our perception of time. As we shall see in Ch. 5, this property is important for quantitative representation of bifurcations and other biological processes which require a necessary departures from conventional notion ofg time. Note the methodological differences with isotime. In the former there is no ordering of the product, while the latter requires the necessary restriction of all products to the right for motion forward in time, and all products to the left for motion backward in time. Thus, on rigorous grounds, the use of isotopic methods is inappropriate whenever the isounit is not time-reversal invariant, $\hat{1}(t, ...) \neq \hat{1}(-t, ...)$ and the use of genotopic methods is necessary with the identifications $\hat{1}^{>} = \hat{1}(t, ...)$ and ${}^{<}\hat{1} = \hat{1}(-t, ...)$.

The last generalization, that of *multivalued forward and backward hypertime* $\{\hat{t}^{>}\} = t \times \{\hat{1}^{>}(t, ...)\}$, $\{{}^{<}\hat{t}\} = t \times \{{}^{<}\hat{1}(t, ...)\}$, is the ultimate, most general possible notion of time which preserves the basic geometric axiom of time, thus being compatible with our perception. It is based on the fact that the current, generally tacit

and are derivable from the *forward hyuperaction*

$$\{\hat{A}^{>}\} = \int_{t_1}^{t_2} \{\hat{L}^{>}\}(t^{>}, \hat{r}^{>}, \hat{v}^{>}), \tag{4.16}$$

where the *forward hyperlagrangian* is a conventional form only properly written in forward hyperspaces (that is, with all variables and products formulated via the preceding hyperstructures).

In addition to all properties of the isonewton and genonewton equations of the preceding sections, the above hypernewton equations have the additional capability of being multivalued, that is, of performing the transition from one single "isoparticle" or "genoparticle" to "multivalued hyperparticles".

More specifically, the mass m in Eq.s (4.15) is *one*, yet the equations describe a *system* of particles

$$\{\hat{m}^{>}\} = \{m_1, m_2, ...\} = \{m \times \hat{1}_1^{>}, m_2 \times \hat{1}_2^{>}, ...\}. \tag{4.17}$$

The latter property signals the crossing of the threshold of applicability in the physical world and the entrance within the arena of theoretical biology. In fact, one single cell can produce a system of cells, but one single particle remains so in physical systems.[11]

[11] Prof. Tepper Gill [Howard University, Washington, D.C.) is studying the representation of *nuclear fission* via the isotopic methods, which is the physical event closest to the multiplication of cells in theoretical biology. However, the preceding physical case can be well studied via *reducible isounits*, that is, isounits given by the *tensorial product* of individual terms

$$\hat{1} = \hat{1}_1 \times \hat{1}_2 ....$$

By comparison, the hyperunits here considered are characterized by a *set*

$$\{\hat{1}\} = \{\hat{1}_1, \hat{1}_2, ...\}.$$

It then follows that the former isounit *is not* multivalued, e.g., because the isoexpectation values yield one single number. Only the latter is. As such, the formulations based on the latter unit are broader than those based on the former. This case also illustrate that the

$$\frac{\{\hat{\partial}>\hat{b}\mu>\}}{\{\hat{\partial}\hat{t}>\}} = \{\omega^{\mu\nu}\} \frac{\{\partial>\}\{\hat{H}>\}}{\{\hat{\partial}\hat{b}^{\nu}>\}}, \tag{4.22b}$$

where

$$\{\omega_{\mu\nu}\} = \omega_{\mu\nu}\{1, 1, ...\}, \quad \{\omega^{\mu\nu}\} = \omega^{\mu\nu}\{1, 1, ...\}, \tag{4.23}$$

in view of the properties

$$\{\partial^{>}\hat{D}^{>}_{\nu} / \hat{\partial}\hat{b}^{\mu>}\} \equiv (\partial D^{\circ}_{\nu} / \partial b^{\mu}), \{1, 1, ...\}, \; D^{\circ} = (p, 0). \tag{4.24}$$

The preservation of the conventional symplectic and Lie structures under the above class of hyperlifting is the most important result of this section. In fact, it confirms the existence of a consistent step-by-step hyperstructural lifting of the entire body of genotopic methods, including the symplectic geometry, and the Lie-Santilli iso- and genotheories (that is, including genoalgebras, genogroups, genosymmetries, genorepresentations, etc.) [9,10]. In turn, this occurrence ensures the axiomatic consistency of the hyperstructural methods here selected, thus its availability for consistent applications.

The latter results include the existence of a simple, yet unique and unambiguous *naive hyperquantization*

$$\{\hat{A}^{>}\} \rightarrow -i\{\hat{1}^{>}\} Ln\{\hat{\psi}^{>}\}, \tag{4.25}$$

under which we have the *forward hyperschrödinger equations*

$$i\frac{\{\partial^{>}\}\{\hat{\psi}^{>}\}}{\{\partial\hat{t}^{>}\}} = i\{\hat{1}_{t}^{>}\}\frac{\partial\{\hat{\psi}^{>}\}}{\partial t} = \{\hat{H}^{>}\}\{>\}\{\hat{\psi}^{>}\} \tag{4.26a}$$

$$\{\hat{p}^{>}_{k}\}\{>\}\{\hat{\psi}^{>}\} = -i\frac{\{\hat{\partial}^{>}\}\{\hat{\psi}^{>}\}}{\{\partial\hat{r}^{k}\}} = -i\{S_{k}^{i}\}\frac{\partial\{\hat{\psi}^{>}\}}{\partial r^{i}}. \tag{4.26b}$$

We finally have the following *forward hyperheisenberg equations*

5: PRELIMINARY APPLICATIONS IN THEORETICAL BIOLOGY

5.1. Outline of main implications

We finally illustrate in this section the applications of isotopic, genotopic and hyperstructural methods to theoretical biology by keeping in mind, as indicated in Sect. 1, that this author is a *theoretical physicist* and, as such, the objective of this section is that of merely *indicating* the possibilities of the new methods. Rigorous and detailed studies in theoretical biology are expected to be the task of interested theoretical biologists.

It appears recommendable to begin with an outline of the *main implications* in the use of isotopic, genotopic and hyperstructural method as necessary guidance for the understanding of the subsequent results.

Along the latter lines, as stated in the abstract, by far the biggest difficulties in the applications of the new methods are due to the limitations of our perception of nature.

To begin, we have our own instinctive *perception* of time and, when investigating any other structure, whether physical, chemical or biological, we assume instinctively that they have our own time. Stated differently, we currently believe that all possible physical, chemical or biological structures everywhere in the Universe evolve with our own time, except for relativistic and gravitational corrections, when applicable.

The first and perhaps most important implication of the new methods can be expressed as follows:

Implications for time: *Biological structures represented with isotopic,*

under the condition[12]

$$(\text{D = Time intervals})^2 \times (\text{ I = Unit of time }) = \text{ invariant }). \qquad (5.2)$$

As an illustration, consider our perception of the very slow growth of plants. Isotopic methods identify the possibility to be resolved by future studies that the plant may well grow at a rate much faster (or much slower) when treated in its own structure.

Suppose that our perception of a certain growth requires a period of time with Euclidean invariant 10,000 sec^2 *when reported to our unit of time* +1 sec. Then, if the internal structure of the plant implies an isotime, say, of 1,000 sec, then the same growth requires only 10 sec,

$$(\text{ D = 10,000 sec}^2) \times (\text{ I}_t = +1 \text{ sec }) \equiv (\hat{\text{D}} = 10 \text{ sec}^2) \times (\hat{\text{I}}_t = 1{,}000 \text{ sec }). \quad (5.3)$$

When genotopic methods are used, we have the emergences of *three* additional notions of time besides that of our perception, as per Sect. 3.2.

When hyperstructural methods are used, *time becomes multidimensional in each of the preceding four possible directions* (Sect. 4.2), thus exiting the boundaries of our intuitional capabilities.

As a final limit of complexity, the DNA code may well imply *four different hypertimes each having infinite dimension.*

The reader should be aware that the above new behaviour of time is structurally beyond Einstein's special and general relativities (but is admitted and quantitatively treatable by our covering isospecial relativity [27–29]) because it may occur for *structures at rest* and in the *absence of gravitation,* thus preventing any applicability of relativistic or gravitational corrections.

The scientific credibility of the above possibilities is established by the

[12] Recall from Sect. 2.2 that the use of the different invariant (Period of time) / (Unit of time) would lead to a different geometry which *is no longer an isotopy,* thus preventing compatibility with our own perception.

"time machines", that is, they are capable of altering time in both the future and the past. Moreover, the alteration is fully causal if treated with isotopic methods because, as shown in Sect. 2, *motion forward in time referred to a positive unit +1 sec is fully equivalent on causal grounds to motion backward in time referred to a negative unit −1 sec.*

Studies of the above causal alteration of time have been conducted at the relativistic level in monograph [10] (which we cannot report here for brevity), and have shown the capability by biological structures under isotopic representation of performing *a closed loop inside the forward light cone,* i.e., the capability of initiating at one point t in space–time, move arbitrarily forward or backward in time and then returning to the original time t.

Almost needless to say, we possess today a rigorous *mathematical representation* of the above behaviour, but we have no clue on how this could possibly be achieved by biological structures.[13]

Additional, equally deep implications are implied by the isotopic, genotopic and hyperstructural methods in regard to our *perception* of space, with predictable greater departures from our intuitional capabilities. The main implications can be expressed as follow:

> **Implications for space:** *When represented with isotopic, genotopic and hyperstructural methods, biological structures have intrinsic space characteristics which are generally different than those of our exterior perception in regard to shape, dimension as well as behaviour in time.*

In essence, we perceive the space characteristics of biological structures via our rather limited three Eustachian tubes which imply the perception of three dimensions represented by the Euclidean space $E(r,\delta,R)$, $r = (x, y, z)$, with metric $\delta = \text{diag.}\,(1, 1, 1)$ over the reals $R(n,+,\times)$. This perception is ultimately reducible to the Cartesian coordinates x, y, z each possessing the *same* unit +1 cm (or equivalent).

[13] In physics the realization predicted for the above "time machine" is that via the use of large positive energies due to matter for motion forward in time and large negative energies due to antimatter for motion backward in time [10]. It is evident that the same interpretation does not apply to biological structures on numerous counts.

The use of the hyperstructures implies further departures from our intuition because each of the four different dimensions becomes multidimensional, the limit of infinite dimensionality not being excluded for the most complex possible biological cases, such as the DNA code.

In reality, the reader should be aware that the dimension of an object can be different for the exterior and the interior geometry even for the case of isotopies, as it occurs under *nondiagonal isounits* (see the end of Sect. 2.2.b). In particular, the exterior observer can perceive a solid, three–dimensional object occupying all three–directions of space, while the internal object can in actuality be *uni–dimensional,* as illustrated by the isotopy (2.2.23)–(2.2.24).

In short, the *decrease* of the internal dimensionality is predicted by the *isotopies,* while the *hyperstructures* generally imply an *increase* of the dimensionality as perceived by us.

It should be stressed that the above novel behaviour of space is beyond Einstein's special and general relativities (but again predicted and quantitatively treatable by our isospecial relativity [37–29]) because it may occur at rest, thus independently from any relativistic correction, and holds without any consideration to curvature, thus in the absence of a gravitational field.

Again, we are merely referring to *unverified predictions* of anomalous space behaviour by our generalized methods, with the understanding that they are expected to be confronted with evidence and resolved in due time.

To express a *personal opinion,* this author is not evidently certain that the isotopic behaviour in space will eventually result to be correct. Nevertheless, this author is convinced that our perception of space is grossly insufficient for quantitative scientific studies of biological structure.

The limitation of our perception of space is illustrated by immersing a *straight* stick in a glass of water and observing that it is *bent* at the water surface. Now, we all know that the stick is straight. Suppose however, that a person is shown for the fist time the straight stick immersed in a glass of water without removing it from the water. That person will be convinced that the stick is bent, contrary to physical reality.[14]

[14] It is intriguing to note that *the stick immersed in water remains straight in the*

a Hamiltonian (otherwise they would be reducible to pure physical processes, as indicated in the Preface). After investigating a number of alternatives, this author submits herein the hypothesis of representing the novel biological effects with generalized *units* of space and time. Alternative representations have been excluded because not compatible with our sensory perception. The characterization of a biological system therefore requires a conventional (Lagrangian or) Hamiltonian representing all actions derivable from a potential V, plus the three-dimensional units of space and the one-dimensional unit of time,

$$H = K + V, \quad \hat{1}_S = \text{diag.}\,(b_x^2, b_y^2, b_z^2), \quad \hat{1}_t = b_t^2, \qquad (a)$$

where: K is a "kinetic-type" term, V is the "potential-type term" and the b's are the *characteristic functions* of the considered cell with any needed functional dependence on time, coordinates, density, internal reactions, etc. It is then evident that the space isounit may describe the actual nonspherical and deformable shape of the cell considered, e.g., a spheroidal ellipsoid with semiaxes $b^\circ_x{}^2, b^\circ_y{}^2, b^\circ_z{}^2$, plus all nonlinear, nonlocal-integral and nonpotential-nonhamiltonian internal effects represented by a factorized function $\Gamma(t, r, \psi, ...)$,

$$\hat{1}_S = \text{diag.}\,(b_x^2, b_y^2, b_z^2) = \hat{1}_S = \text{diag.}\,(b^\circ_x{}^2, b^\circ_y{}^2, b^\circ_z{}^2) \times \Gamma(t, r, \psi, ...),$$

with more complex shapes and internal effects representable with nondiagonal realizations. The time isounit $\hat{1}_t$ generally provides a geometrization of the remaining features, such as density, chemical composition, etc. In general, the above space-time units are not invariant under time-reversal, in which case they technically are *genounits of space and time*. As such, they provide an axiomatic characterization of the irreversibility of the evolution of the cell considered by assuming, e.g., for forward and backward units and their isoduals

$$\hat{1}_a^> = \hat{1}_a(t, ...), \quad {}^<\hat{1}_a = \hat{1}(-t, ...), \quad \hat{1}^{>d} = -\hat{1}^>, \quad {}^<\hat{1}_a = -{}^<\hat{1}_a, \quad a = s, t.$$

This sets the foundation for the representation of all possible four time arrows which are all admitted by our sensory perception (Fig. 4.2), thus permitting quantitative representations of bifurcations and other complex events necessarily requiring negative values of time (see Fig. 5.9), which representation would be merely impossible via conventional methods. The emergence of the multivalued hyperstructures is simply unavoidable if one reflects a moment on the fact that the cells of a complex organism are generally different, thus requiring different units of space and time. Again, the sole representation of the latter occurrence

The above implications have been studied in Sect. 2.2, see., e.g., Fig. 2.2, and they are essentially yet another consequence of the possible alteration of the basic units. Consider again the isobox, and suppose that the internal structure is in condition to alter the unit of the x–axis (only) in an isotopic way. Then the shape remains unchanged to the outside observer. Yet, *the box will move along the x–axis for the outside observer without any application of Newtonian forces, while no motion occurs for an interior observer.* This is due to the fact, studied in Sect. 2.2, that the isotopic change of the unit along the x–axis has altered the distance of the (center of the) isobox from the origin. Motion with respect to the external observer is then consequential.

Even deeper implications are expected in locomotion via the use of the genotopic and hyperstructural methods, to such an extent to coss again the boundaries of our imagination.

5.2. Apparent isotopic structure of sea shells

We now apply the simplest possible generalized methods, those of isotopic type, to one of the simplest possible biological problems, the quantitative representation of the growth of sea shells, which has been studied in details in the recent monographs by Illert and Santilli [45].

The first point to clarify is that *the infinitely possible **shapes** of sea shells can all be well represented in the conventional Euclidean geometry without any need of any generalization of any kind* As an example, consider the sea shell of the *nipponites mirabili* in by Illert [45], Eq.s (3.27), p. 90)

$$x = a\, e^{\alpha \phi} [\, 1 + e^{\phi} \cos (2 \gamma \phi)] \cos \phi ,$$

$$y = a\, e^{\alpha \phi} [\, 1 + e^{\phi} \cos (2 \gamma \phi)] \sin \phi ,$$

$$z = b\, e^{\beta \phi} \sin (\gamma \phi) , \qquad (5.8)$$

cit.] noted first that the strict imposition of the Euclidean axioms in three–dimensional space *do not* permit such a realistic representation. Santilli [45] then proved that the desired quantitative representation can be obtained via the use of the isoeuclidean geometry.

The argument can be summarized as follows. Let us consider first the *assumption* that the growth in time of the sea shells can be *exactly and quantitatively* represented via the conventional Euclidean geometry in three dimensions, as used for its shape. This representation requires the necessary enlargement of the pure Euclidean space to the Kronecker product

$$S(t, \xi, \psi) = E(t,R_t) \times E(\xi,\delta,R) \times E(\psi,\delta, R)\,, \tag{5.9}$$

where: $\xi(\phi = \{\xi_x(\phi), \xi_y(\phi), \xi_z(\phi)\})$ is a vector–value function representing the shape, ϕ is a characteristic angle of growth of sea shells with simple structure, $\psi = d\xi/dt$ represents the rate of growth, and $E(\psi,\delta,R)$ is the *tangent space* to $E(r,\delta,R)$. The underlying total unit is then given by the familiar seven dimensional quantity

$$I_{tot} = 1 \times \text{diag.}\,(1, 1, 1) \times \text{diag.}\,(1, 1, 1). \tag{5.10}$$

The imposition of the Euclidean character of the growth identifies the applicable methods in a rather rigid way. In fact, the methods must be of *Lagrangian* type (Sect. 2.2) (with equivalent Hamiltonian form here ignored for brevity) and, to avoid the impression of a scientific derivation, must be *restricted to verify the Euclidean axioms.* In particular, the product among two vectors must be Euclidean, to avoid inconsistency with the basic assumption, e.g.,

$$\xi^2 = \xi \times \xi = \xi_x\, \xi_x + \xi_y\, \xi_y + \xi_z\, \xi_z\,, \tag{5.11}$$

where $\times$ is the ordinary scalar product; the distance among two vectors must be the familiar form

$$D_{Euclidean} = [\,(\xi_{1x} - \xi_{2x})^2 + (\xi_{1y} - \xi_{2y})^2 + (\,\xi_{1z} - \xi_{2z})^2\,\}^{1/2}, \tag{5.12}$$

The applicable *isolagrangian* is then given by

$$\hat{L} = [\, K_1 \, (\frac{\hat{\partial}\xi}{\hat{\partial}t} \hat{\times} \frac{\hat{\partial}\xi}{\hat{\partial}t})^{\hat{n}} + K_2 \, (\frac{\hat{\partial}\xi}{\hat{\partial}t} \hat{\times} \xi)^{\hat{m}} + K_3 (\xi \hat{\times} \xi)^{\hat{p}} \,]\hat{1} \,, \qquad (5.18)$$

with related *isolagrange equation* (see Sect. 2.3.c) which are ignored here for brevity.

The following property is evident from the completely unrestricted functional dependence of the isounits of space and time as well as their not necessarily diagonal character.

Lemma 5.1 (see Santilli in [45]): *The isoeuclidean geometry is "directly universal" for the representation of the growth of sea shells, i.e., it is capable of representing the growth of all infinitely possible shapes (universality), directly in the frames of the observer (direct universality).*

As a simple example, the Lagrangian of ref. [45], Eq. (1.36), p. 25, i.e.,

$$L = K_1 \, \frac{d\xi}{d\phi} e^{\Omega\phi} \, \frac{d\xi}{d\phi} + K_2 \xi e^{\Omega\phi} \xi \qquad (5.19)$$

where Ω is a characteristic function of the particular sea shell considered, is a clear particular case of the isolagrangian with simple scalar isounit

$$\hat{1}_t = 1, \quad \hat{1} = e^{-\Theta\phi} \, \text{diag.}\,(1, 1, 1) \,. \qquad (5.20)$$

The isorepresentation of all other models studied by Illert then follows and it is left as an exercise for readers interested in learning isotopic methods.

A few comments are now in order. First, note that the last simplest possible example is representable with the conventional time unit and the isotopy only of space. It is nevertheless easy to see the *necessity* of lifting also the time unit, which is expected whenever the three isoscalar products in the isolagrangian (5.18) must have different isostructures.

mere mathematical formality, because it permits the *reduction of sea shells with arbitrary shapes to an isoshape possessing the exact isorotational symmetry* $\hat{O}(3)$ of App. C.

SEA SHELLS AS "TIME MACHINES"

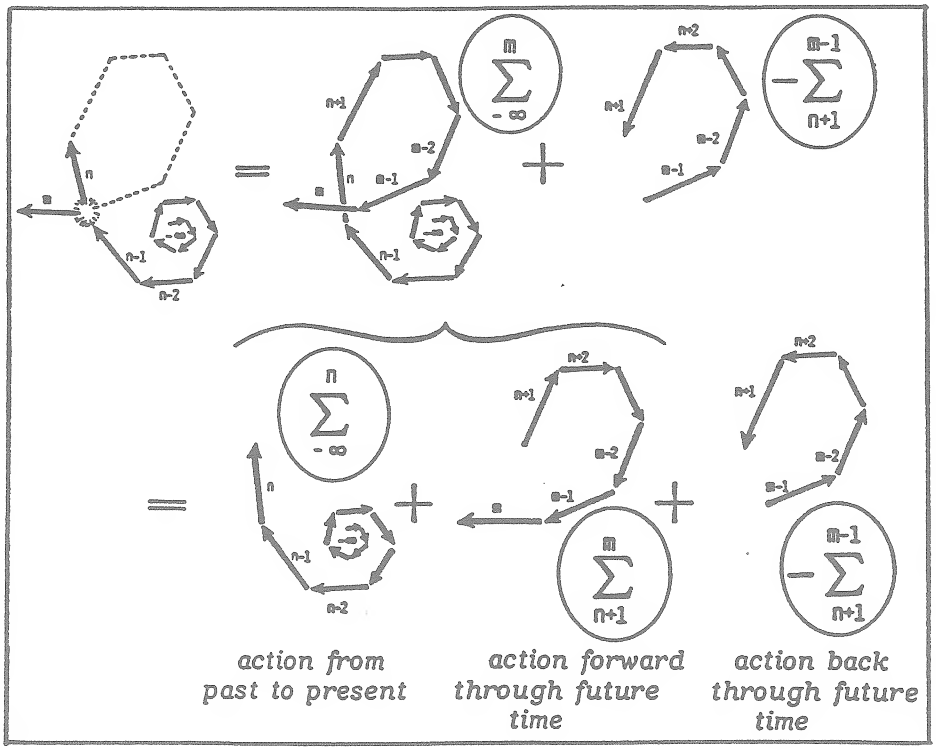


FIGURE 5.3. A reproduction of the figure by Illert in [45], p. 95, on sea shell bifurcations. Their *quantitative* representation of such bifurcations, e.g., in a form which can be visualized in a computer, requires three contributions: 1) Action forward from past to present time, 2) action forward through future time, and 3) action backward from future time. It appears evident that such a behaviour is

HYPERSTRUCTURAL CHARACTER OF SEA SHELLS

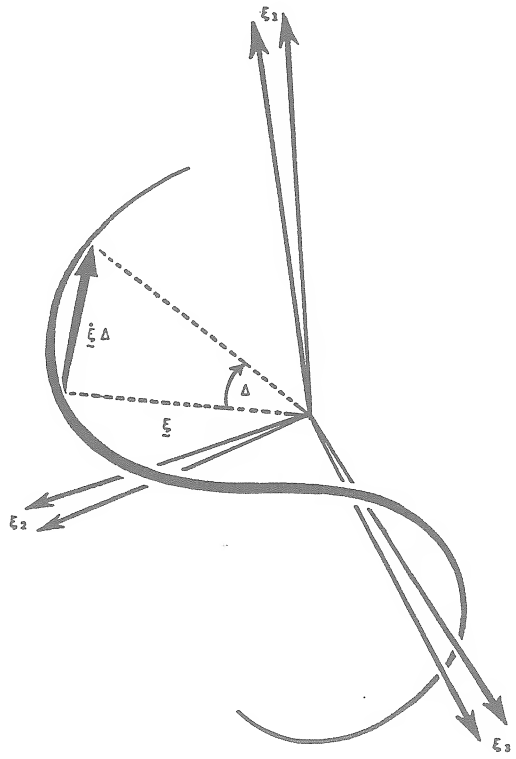


FIGURE 5.4: A reproduction of the table by Illert in [45], p. 11, illustrating the need of doubling the number of dimension for a quantitative interpretation of sea shells growth, i.e., passing from three to six dimensions of space. A deeper study indicates that this occurrence requires in effect the doubling of the dimension of *each axis*, thus yielding precisely *the hyperstructure with two-dimensional hyperunits*, resulting in the the total 14-dimensional hyperunit

$$\{\hat{1}_{tot}\} = = \{\hat{1}_t\} \times \{\hat{1}_\xi\} \times \{\hat{1}_\psi\} = \{\hat{1}_{t1}, \hat{1}_{t2}\} \times \{\hat{1}_{\xi 1}, \hat{1}_{\xi 2}\} \times \{\hat{1}_{\psi 1}, \hat{1}_{\psi 2}\},$$

with corresponding 14-dimensional hyperspace

$$x\,T_x\,x + y\,T_y\,y + z\,T_z\,z =$$

$$= r^2 (B_{22}^2\, B_{11}^2\, \text{isosin}^2\,\hat{\theta}\,\hat{\cos}^2\,\hat{\phi} + B_{22}^2\, B_{12}^2\, \text{isosin}^2\,\hat{\theta}\ \text{isosin}^2\,\hat{\phi} +$$

$$+ B_{21}^2\, \text{isocos}^2\,\hat{\theta}\,) = r^2 . \qquad (5.27)$$

Similar representations hold for all other possible shapes of sea shells owing to the direct universality of the isoeuclidean geometry. The above examples confirm the geometric unification of all possible sea shells into the isosphere.

Such a geometric unification permits the identification of the *universal symmetry of sea shells as being the isorotational symmetry* $\hat{O}(3)$ of App. C. In turn, the application of the isorotations is particularly intriguing because it permits the study of interconnections between sea shells which simply cannot be studied via the conventional rotational symmetry. This aspect will be studied in a separate work.

In closing this subsection, this author would like to disclose that he did not expect the emergence in the above relatively "simple" problem of the most general possible formulations studied in this memoir, the hyperstructural methods of Class V, and would like to report here his surprise.

In fact, in the main text of this subsection we have used the isotopic formulations of Class I ($\hat{1} > 0$). The bifurcation of sea shells have then indicated, first, the need of formulations of Class III ($\hat{1} >0$ or < 0), then the need of genotopic theories for a more accurate axiomatization of time, and then the genotopic theories of Class IV (admitting the zeros $\hat{1} = 0$). Moreover, the multidimensional and discrete character of the growth of sea shells has indicated the need of the most general possible methods studied in this memoir, the hyperstructures of Class V with discrete isounits. Finally, the possibility of reducing arbitrary shapes of sea shells to the perfect sphere in isospace illustrates that *the totality of the classical methods studied in this memoir enter, whether directly or indirectly, in the quantitative representation of the relatively "simple" growth of sea shells.*

nonhamiltonian character of the interactions due to the wave-overlapping of the wavepackets of the electrons and positrons at distances of the order of their classical radius. In fact, these nonlocal interactions can be approximated via a Hulten potential which behaves precisely like a Coulomb potential at sort distances. But the nonlocal interactions resulted to be stronger than the Coulomb one at short distances. It therefore followed that the nonlocal interactions would dominate over the Coulomb one in the structure $\pi^\circ = (e^-_\uparrow, e^+_\downarrow)_{HM}$.

The main hypothesis submitted in memoir [6a], Sect. 5, was therefore that the isotopies provide a quantitative representation of the transition

$$\text{Positronium} = (e^-_\uparrow, e^+_\downarrow)_{QM} \quad \rightarrow \quad \pi^\circ = (e^-_\uparrow, e^+_\downarrow)_{HM} . \qquad (5.28)$$

In a more recent contribution, Animalu [46] noted that the above nonlocal interactions at short distances are so strong to provide a quantitative representation of the Cooper pair in superconductivity

$$\text{Cooper Pair} = (e^-_\uparrow, e^-_\downarrow)_{HM} . \qquad (5.29)$$

The latter idea was then studied in details by Animalu and Santilli [43] and shown to be fully compatible with experimental evidence in superconductivity.

The application of the above lines to the valence is done in this section apparently for the first time. By assuming here a technical knowledge of hadronic mechanics, the main results can be outlined as follows.

Consider one electron with charge $-e$, spin up and wavefunction $\psi_\uparrow$ in the field of another electron with the same charge $-e$, spin down and wavefunction $\psi_\downarrow$ considered as external. Its Schrödinger equation is given by the familiar expression

$$H_{Coul.}\, \psi(t, r) = \{ \frac{1}{2m} p_k p^k + \frac{e^2}{r} \}\, \psi_\uparrow(t, r) = E_o\, \psi_\uparrow(t, r) ,$$

$$p_k\, \psi_\uparrow(t, r) = - i\, \partial_k\, \psi_\uparrow(t, r) , \qquad (5.30)$$

where m is the electron rest mass. The above equation represents *repulsion,* as well

$$\hat{p}_k \, T \, \hat{\psi}_\uparrow(t, r) = - i \hat{1}_k^{\,1} \partial_i \hat{\psi}_\uparrow(t, r) . \tag{5.32}$$

In order to achieve a form of the model which can be confronted with experimental data, we need an explicit expression of the isounit $\hat{1}$. Among various possibilities, we here select the simplest possible form called *Animalu's isounit* [46], which we write here in the form

$$\hat{1} = e^{- < \hat{\psi}_\uparrow | \hat{\psi}_\downarrow > \psi_\uparrow / \hat{\psi}_\uparrow} \approx 1 - < \hat{\psi}_\uparrow | \hat{\psi}_\downarrow > \psi_\uparrow / \hat{\psi}_\uparrow + ...$$

$$\hat{1} = e^{+ < \hat{\psi}_\uparrow | \hat{\psi}_\downarrow > \psi_\uparrow / \hat{\psi}_\uparrow} \approx 1 + < \hat{\psi}_\uparrow | \hat{\psi}_\downarrow > \psi_\uparrow / \hat{\psi}_\uparrow + ... \tag{5.33}$$

under which Eq.s (5.32) can be written

$$\{ \frac{1}{2m} \hat{p}_k \, T \, \hat{p}^k T \, \hat{\psi}_\uparrow - (z - 1) \frac{e^2}{r} \hat{\psi}_\uparrow - z \frac{e^2}{r} < \hat{\psi}_\uparrow | \hat{\psi}_\downarrow > (\psi_\uparrow / \hat{\psi}_\uparrow) \hat{\psi}_\uparrow = E \, \hat{\psi}_\uparrow \tag{5.34}$$

Now, it is well known from quantum mechanics that the radial part of $\psi_\uparrow$ in the ground state [L = 0) behaves as

$$\psi_\uparrow(r) \approx A \, e^{-r/R} , \tag{5.35}$$

where A is (approximately) constant and R is the coherence length of the valence pair. The radial solution for $\hat{\psi}_\uparrow$ also in the ground state is known from Eq.s (5.1.21), p. 837, ref. [6b] to behave as

$$\hat{\psi}_\uparrow(r) \approx B(1 - e^{-r/R}) / r, \tag{5.36}$$

where B is also approximately a constant. The last term in the l.h.s. of Eq. (5.34) therefore behaves like a Hulten potential

$$V_0 \, e^{-r/R}/(1 - e^{-r/R}), \qquad V_0 = e^2 < \hat{\psi}_\uparrow | \hat{\psi}_\downarrow > . \tag{5.37}$$

After substituting the expression for the isomomentum, the radial

where c_0 is the speed of light in vacuum, for which

$$V = 2 k_1 k_2^2 \hbar c_0 / R , \tag{5.43}$$

and the total energy of the hadronic state becomes in the ground state (which occurs for n = 1 for the Hulten potential)

$$E_{tot,\, \pi^\circ} = 2 k_1 [1 - (k_2 - 1)^2 / 4] \hbar c_0 / R = 2k_1 (1 - \epsilon^2) \hbar c_0 / R . \tag{5.44}$$

Isotopic model of the π°. The use of the total energy of the π° (135 MeV), its charge radius ($R \approx 10^{-13}$ cm) and its meanlife ($\tau \approx 10^{-16}$ sec), then yields the values (Eq.s (5.1.33), p. 840, ref. [6b])

$$k_1 = 0.34 , \quad \epsilon = 4.27 \times 10^{-2} , \quad k_2 = 1 + 8.54 \times 10^{-2} > 1 . \tag{5.45}$$

The experimental verification of the model is established by the fact that it describes the totality of the characteristics of the π°, such as rest energy, charge radius, mean life, charge, charge parity, magnetic moment, etc. [6b]. In addition, the model represents the decay with the *lowest* rate, $\pi^\circ \to = e^+ + e^-$ as a *tunnel effect of the constituents.*

The above *structure model* of the π° identifies its constituents with massive physical particles which are produced free in the spontaneous decays with the lowest rate. Moreover, the representation occurs in our space–time *only* without any recourse to internal unitary symmetries. Nevertheless, when the *separate problem of classification* is considered, the addition of unitary internal symmetries does permit the achievement of compatibility with the unitary model.

The above model has been extended to all unstable mesons and, more recently, to unstable baryons, including the first representation on scientific records of the synthesis of the neutron as it initially occurs in stars, from protons and electrons *only* (because stars at their formation are notoriously composed

Isotopic model of the Cooper pair. Animalu and Santilli [43] identified the solution of Eq.s (.5.38) for the Cooper pair by introducing the parameters

$$k_1 = \epsilon_F R / \hbar c_o, \qquad k_2 = KR / \epsilon_F, \tag{5.46}$$

where ϵ_F is the iso–Fermi energy of the electron (that for hadronic mechanics).

The total energy of the Cooper pair in the ground state is then given by

$$E_{Tot,\ Cooper\ pair} = 2 k_1 [\, 1 - (k_2 - 1)^2 / 4 \,] \hbar c_o / R \approx k_2 T_c / \theta_D , \tag{5.47}$$

where θ_D is the Debye temperature.

Several numerical examples were considered in ref. [43]. The use of experimental data for aluminum ($\theta_D = 428°K$, $\epsilon_F = 11.6$ eV, $T_c = 1.18°K$) yields the values

$$k_1 = 94 , \qquad k_2 = 1.6 \times 10^{-3} < 1 . \tag{5.48}$$

For the case of $YBa_2Cu_3O_{6-\chi}$ the model yields [43]

$$k_1 = 1.3\, z^{-1/2} \times 10^{-4}, \qquad k_2 = 1.0\, z^{1/2} > 1 , \tag{5.49}$$

where the effective valence $z = 2(7 - \chi)/3$ varies from a minimum of $z = 4.66$ for $YBa_2Cu_3O_{6.96}$ ($T_c = 91°K$) to a maximum of $z = 4.33$ for $YBa_2Cu_3O_{6.5}$ ($T_c = 20°K$). The general expression predicted by hadronic mechanics for $YBa_2Cu_3O_{6-\chi}$ is given by (Eq. (5.15), p. 373, ref. [43])

$$T_C = 367.3 \times z \times e^{-\,13.6\ /\ z} . \tag{5.50}$$

The above model of the Cooper pair did indeed was proved to be fully compatible with experimental data on superconductivity (Fig. 5.5), thus establishing the scientific credibility of the entire approach, including its application to the structure of the π° meson as well as to the bonding of atoms in molecules.

It should be indicated that the above isotopic model of the Cooper pair has

(solid dots).

NUMERICAL VERIFICATIONS

Table 1. $YBa_2Cu_{3-x}Mn_xO_y$
(After N.L. Saini *et al.*, Int. J. Mod. Phys. B6, 3515 (1992)

x	y	z	T_c (theory)	T_c (expt.)
0.00	6.92	4.613	88.9	91
0.03	6.88	4.541	83.5	86.6
0.09	6.87	4.447	76.7	79.0
0.15	6.91	4.387	72.6	75.0
0.21	6.92	4.312	67.6	72.0
0.30	6.95	4.212	61.3	67.0

Note: T_c (theory) $= 367.3z\exp(-13.6/z)$, where the effect of replacing Cu_3 by $Cu_{3-x}Mn_x$ is obtained by replacing 3 by $(3-x)+2x=3+x$, which lowers the effective valence (z) on Cu^{2+} ions to $z=2y/(3+x)$.

Table 2. $GdBa_2(Cu_{1-x}Ni_x)_3O_{7-\delta}$
(After, Chin Lin *et al.*, Phys. Rev. B42, 2554 (1990))

x	$y=7-\delta$	z	T_c (theory)	T_c (expt.)
0.000	6.96	4.640	91.0	91
0.025	6.96	4.527	82.4	79
0.050	6.96	4.419	74.8	71
0.075	6.96	4.316	67.9	65

Note: T_c(theory) $= 367.3z\exp(-13.6/z)$, $z=2y/3(1+x)$ as discussed in Table 1.

Table 3. $GdBa_2(Cu_{1-x}Zn_x)_3O_{7-\delta}$
(After, Chin Lin *et al.*, Phys. Rev. B42, 2554 (1990))

x	$y=7-\delta$	z	T_c (theory)	T_c (expt.)
0.000	6.96	4.640	91.0	91
0.025	6.96	4.309	67.4	54
0.050	6.96	4.009	49.0	37
0.075	6.96	3.737	36.1	35

FIGURE 5.6. A reproduction of the table of p. 379, ref. [43], illustrating the experimental verification of the prediction of the isotopic treatment of nonlocality in superconductivity from viewpoints different than that of Fig. .5.5.

identically as a particular case, although *without* attraction.

A comparison of the above nonlocal isotopic model of atomic bonding into molecules with conventional quantum mechanical models, such as the *density functional model, the Kohm–Sham model,* and others [1,2,3] is now in order.

The most visible and fundamental difference is that *the isotopic model does indeed achieve an attractive interaction among individual pairs of atoms, while quantum mechanical models achieve attraction on statistical/density matrix form.* As it has been the case for the structure of the π° and that of the Cooper pair, we therefore expect the isotopic model to have greater predictive capacities than conventional models.

We should also indicate that the isotopic lifting of the density–functional models, here called *isodensity–functional model,*[15] is quite simple and it is expected to be significant, because preserving all conventional characteristics *plus* representing nonlinear, nonlocal and nonhamiltonian effects.

With reference to the outline of hadronic mechanics in Sect. 2.3.F, the fundamental step of the density–isofunctional method is the selection of the explicit form of the isounit $\hat{1}$ and isotopic element $\hat{T}$, such as Eq. (5.33), while the Hamiltonian remains the conventional one, $H = K + V$, where K is the usual kinetic energy and V is the sum of the usual terms (see [1,2,3] for details), although these terms must now be formulated in isospace, i.e., with all isotopic product ATB; the wavefunction (written $\hat{\psi}$ to distinguish it from the conventional symbol ψ because of different explicit forms) is now an isostate of an isohilbert space $\hat{\mathcal{H}}$ with isonormalization

$$< \hat{\psi} \mid T \mid \hat{\psi} > \; = \; 1 \; ; \tag{5.52}$$

the isoexpectation values of an observable $\hat{A}$ are now given by

$$< \hat{A} > \; = \; < \hat{\psi} \mid T \, A \, T \mid \hat{\psi} > \; ; \tag{5.53}$$

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[15] The mathematically correct formulation should be that of the *isodensity isofunctional.* In the following we ignore the isofunctional character for simplicity, and also because it does not imply different numerical results.

alteration of virtually all quantities and expression to account for nonlinear, nonlocal and nonhamiltonian effects.

One should however keep in mind that the latter effects are non–null only under appreciable overlapping of the wave packets. Therefore, *the isodensity functional model coincides with the conventional model everywhere except at distances of the order of the coherence length of the valence electrons.*

Other particle–atomic–molecular effects. Essentially the same mechanism studied for the Cooper pair is applicable, with due adaptations, to a variety of other effects which generally are outside the representational capabilities of quantum mechanics, e.g., because prevented by a Coulomb barrier.

In all these effects we can distinguish the following three phases:

PHASE I: Nonlinearity. This is the mechanism which activates effects beyond quantum mechanics. Recall that, according to our main assumption stated beginning with the Preface, quantum mechanics is assumed to be *exactly* valid for all mutual distances of particles bigger than their coherence wavelength. The effects here considered therefore need a *trigger mechanism* which brings the particle considered at mutual distances smaller than said coherence wavelength, and it is generally represented via *nonlinearity in the wavefunctions.*[16] As an illustration, in the Cooper pair the trigger mechanism is given by the Cuprate ions. In other cases, the trigger mechanism can be given by simple osmotic pressures and the like.

PHASE II: Nonlocality. Once the particles considered have been broth under conditions of appreciable overlapping of their wavepackets, we have the emergence of attractive or repulsive nonlocal effects which can be maximized via the use of axiomatically consistent methods, such as the isotopies, genotopies and hyperstructures. In particular, nonlocal effects are attractive in singlet coupling

[16] We assume the reader is now familiar with the fact that nonlinearity is sufficient alone to require a suitable generalization of quantum mechanics. In fact, *conventional nonlinear quantum mechanical models violate the superposition and other principles thus being structurally unable to represent composite systems such as a biological structure.* The representation of nonlinear effects via the isotopies of quantum mechanics (hadronic mechanics [10]) permits the regaining of the superposition and other principles, thus offering the foundations for rigorous quantitative treatments.